On the Nature of Turbulence in a Stratified Fluid. Part I: The Energetics of Mixing

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ABSTRACT

The definition of the flux Richardson number $R_f$ is generalized to be the ratio of the turbulent buoyancy flux $b$ to the net turbulent mechanical energy $m$ available from all sources. For mechanically energized turbulence where turbulent kinetic energy is used to sustain an upward buoyancy flux ($b > 0$), it is shown the magnitude of $R_f$ is quantitatively determined by the location of the event in the $F_r$--$Re_T$ diagram, where $F_r$ and $Re_T$ are the local instantaneous overturn Froude number and Reynolds number. In this parameter space, the value of $R_f$ varies between 0 and 0.20 for a fluid with Prandtl number greater than one, and between 0 and 0.15 for a fluid with a Prandtl number less than one.

For turbulence sustained by a negative buoyancy flux ($b < 0$), such as penetrative convection in a cooling surface layer, it is shown that the flux Richardson number $R_f$ is a function of depth below the surface; $R_f^{-1}$ varies between 0.55 at the surface and $-\infty$ towards the base of the surface layer where the buoyancy flux vanishes. This result may again be interpreted in terms of location in the $F_r$--$Re_T$ diagram.

Finally, it is shown that once the value of $R_f$ is known the vertical buoyancy flux may be evaluated directly without recourse to a turbulence model.

1. Introduction

The nature of the turbulence in a mechanically generated turbulent flow leading to a mixing event in a stratified fluid is most clearly visualized by an examination of the Froude and Reynolds numbers characterizing the turbulent motion and the ambient stratification (Gibson 1980, 1982, 1986, 1987a, 1987b; Imberger and Boashash 1986; Luketina and Imberger 1989). If $N$ denotes the buoyancy frequency of the background stratification, $u$ the rms velocity of the turbulent motions, and $L_C$ the centered displacement scale, a measure of the scale of the observed overturns or inversions actually registered in the density profile (see Imberger and Boashash 1986), then the energy bearing eddies are characterized by the overturn Froude number

$$ F_r = \frac{u}{NL_C}. $$

Since $L_C$ is the scale of the most energetic overturns, Luketina and Imberger (1989) postulated that the rms velocity scale $u$ is given by

$$ u \sim (\epsilon L_C)^{1/3} $$

where $\epsilon$ is the rate of dissipation of turbulent kinetic energy.

This assumption can be tested with the laboratory results in a stratified water tunnel facility reported by Stillinger et al. (1983), Itsweire et al. (1986) (hereafter denoted by SHV and IHV, respectively) with turbulence generated by a grid in salt stratified water; by Rohr et al. (1988, hereafter denoted by RIHV) with turbulence generated by both a grid and a mean shear in salt stratified water; and with data from the recent work by Lienhard and Van Atta (1990, hereafter denoted by LV) with turbulence generated by a grid in a temperature stratified air tunnel. The energy cascade process described by (2) should be independent of $F_r$ and this is indeed the case as can be seen in Fig. 1 which shows an excellent correlation between the rms velocity $u$ and $(\epsilon L_f)^{1/3}$. The quantity $L_f$ is measured in the above experiments from the rms density fluctuations and the known background density gradient and differs from $L_C$ by a scale factor very close to one (Luketina, personal communication). The implication of the relation in (2) is that the small-scale motions, constituting the high wavenumber components of the turbulence, keep pace in their development with the energy input at larger scales (see also Broadwell and Breidenthal 1982) during the energetic stages of the grid generated turbulence.

Substituting (2) into (1) yields alternative definitions of the Froude number as (Luketina and Imberger 1989)

$$ F_r = \frac{u}{NL_C} = \left( \frac{\Gamma}{N} \right) = \left( \frac{\epsilon}{N^3 L_C^2} \right)^{1/3} = \left( \frac{L_R}{L_C} \right)^{2/3} $$

where the rate of strain of the large-scale fluctuations $\Gamma = (u/L_C)$, and the Oszmidov scale $L_R = (\epsilon/N^3)^{1/2}$.

If we note that $N^2 L_C^{-2}$ is the potential energy per unit
mass and $N$ the gravitational adjustment time scale, then the third expression in (3) indicates that $\text{Fr}_T$ is interpreted as the one-third power of the ratio between the dissipation of TKE and the maximum rate of release of potential energy locked up in the unstable signatures of the density profiles. If $\text{Fr}_T > 1$, the turbulence is energetic with dissipation exceeding the available potential energy (APE) contained in the measured overturn events. Dissipation must, therefore, be balanced by either the local production of turbulent kinetic energy from the mean shear or by the importation of turbulent kinetic energy. Conversely, if $\text{Fr}_T < 1$ the dissipation is less than the APE contained in the unstable portions of the water column. This implies that the potential energy of the motions is predominantly balanced by the kinetic energy fluctuations, as in the case of internal waves or intrusions.

The energy bearing eddies are also characterized by the overturn Reynolds number

$$\text{Re}_T = \frac{u L_C}{\nu}$$

and using (2), this may be written as

$$\text{Re}_T = \frac{u L_C}{\nu} = \left( \frac{u L_C}{\nu} \right)^{4/3} = \left( \frac{L_R}{L_K} \right)^{4/3}$$

where the Kolmogorov scale $L_K = (\nu^3 / \epsilon)^{1/4}$.

In the preceding, three fundamental length scales—$L_C$, $L_R$ and $L_K$—were introduced. The ratios ($L_C/L_R$) and ($L_C/L_K$) can be used to define two dimensionless numbers, the Froude number and the Reynolds number, respectively. The other possible ratio of length scale is ($L_R/L_K$) and this introduced a third possible dimensionless number.

The small-scale Froude number was defined by Imberger and Boashash (1986) as

$$\text{Fr}_y = \left( \frac{\gamma}{N} \right) = \left( \frac{\epsilon}{\nu N^2} \right)^{1/2} = \left( \frac{L_R}{L_K} \right)^{2/3}$$

where $\gamma$ is the rate of strain of the small-scale fluctuations or high wavenumber components of the turbulence defined by:

$$\epsilon = \nu \gamma^2.$$ 

Since there are only three independent length scales, and the dimensionless numbers can all be defined in terms of the ratio of any two of these length scales, the three dimensionless numbers are not independent and are related by

$$\text{Fr}_T = \left( \frac{1}{\text{Re}_T} \right)^{1/2} \text{Fr}_y.$$ 

Equation (8) indicates that the large- and small-scale Froude numbers are connected by the Reynolds number of the overturn motions and the specification of the two parameters $\text{Fr}_T$ and $\text{Re}_T$ is sufficient to characterize the turbulence.

The state of turbulence in a stratified fluid may thus be inferred from the $\text{Fr}_T$-$\text{Re}_T$ diagram. The buoyancy frequency $N$ is not defined either for penetrative convection or within large unstable overturn events. For this reason the above formulations for $\text{Fr}_T$ and $\text{Re}_T$ are generalized in the next section and written in terms of an effective gravitational constant $g'$ instead of $N$.

Gibson (1987b and previous articles) has implied that the $\text{Fr}_T$-$\text{Re}_T$ diagram may be interpreted as an activity diagram where the word “activity” implies the diagram can be used to describe the evolution of the turbulence. While this may be true in certain circumstances, the $\text{Fr}_T$-$\text{Re}_T$ diagram is really only a reflection of the status of the current energy budget of the turbulence as documented by a particular microstructure profile (see Part 2, Imberger and Ivey 1991).

2. The energetics of mixing

The full turbulent kinetic energy equation reads (Tennekes and Lumley 1972)

$$- \frac{\partial}{\partial t} \left( \frac{u_i' u_j'}{2} \right) - \frac{\partial}{\partial x_i} \left( u_i' u_j' \right) - \frac{\partial}{\partial x_j} \left( \frac{1}{\rho_0} u_i' p' \right)$$

$$- \bar{u}_j \frac{\partial}{\partial x_i} \left( \frac{u_i' u_j'}{2} \right) - u_i' u_j' \frac{\partial \bar{u}_i}{\partial x_j} = \frac{g}{\rho_0} u_j' \bar{p}' + \epsilon$$

i.e.,

$$m = m_1 + m_2 + m_3 + m_4 + m_5 = b + \epsilon$$

where $x_i$ are the coordinate axes with direction 3 vertically upward, $b$ is the buoyancy flux, $\epsilon$ is the rate of dissipation of turbulent kinetic energy, and $m$ is the net mechanical energy required (or available) to sustain the turbulent motions.
Now define a generalized flux Richardson number as

\[ R_f = \frac{b}{m} = \frac{1}{1 + \left( \epsilon / b \right)} \]  

(10)

and following Luketina and Imberger (1989), introduce the correlation coefficient

\[ R_{pw} = \frac{\langle \rho' \bar{w}' \rangle}{\langle \bar{p} \rangle \langle \bar{w} \rangle} \]  

(11)

where \( u'_i = w' \) and \( \bar{p} \) and \( \bar{w} \) are the rms vertical density and velocity fluctuations, respectively. Thus

\[ R_f = 1 \left[ 1 + \frac{\epsilon}{(g'/\rho_0)\bar{p}\bar{w} R_{pw}} \right] \]  

(12)

using

\[ g' = \frac{g}{\rho_0} \bar{p} \]  

(13)

we can introduce a generalized definition of the overturn Froude number as

\[ Fr_T = \left( \frac{\epsilon}{g' \bar{w}} \right)^{1/2} \]  

(14)

which can be derived from (1) using \( g' = N^2 L_C \) and \( \bar{w} \sim (\epsilon L_C)^{1/3} \).

Using (14) Eq. (12) becomes

\[ R_f = 1 \left[ 1 + \frac{Fr_T^2}{R_{pw}} \right] \]  

(15)

Linden (1979) and Rohr et al. (1984) have argued on physical grounds that \( R_f \) should be a function of an overall stability of the flow, and the expression in (15) is a quantitative prediction of the form of this dependence. The result is quite general, and is equally valid for such differing flows as a turbulent shear flow or for penetrative convection, for example. In all cases, once the values of \( R_f \) and \( \epsilon \) are known (10) may be used directly to calculate the buoyancy flux \( b \) and the net mechanical energy flux \( m \).

Using the relations above, we can also define generalized forms of both the Froude and Reynolds numbers as

\[ Fr_T = \left( \frac{\epsilon}{g'^{3/2} L_C^{1/2}} \right)^{1/3} \]; \]

\[ Re_T = \left( \frac{\epsilon^{1/3} L_C^{4/3}}{\nu} \right); \quad Fr_T = \left( \frac{\epsilon L_C}{\nu g'} \right)^{1/2}. \]

The small-scale Froude number \( Fr_T \) retains the physical interpretation of the ratio of the rate of strain \( (\epsilon/\nu)^{1/2} \) of the small-scale fluctuations to the rate of adjustment by buoyancy \( g'/L_C \) of the largest unstable signatures in the profile.

3. Turbulence generated by grids or mean shear flows \( (b > 0) \)

The prediction in (15) can be tested with the extensive laboratory results reported by SHV, IHV and RIHV in a stratified water tunnel and the laboratory results of LV for a stratified air tunnel. In the experiments of SHV, IHV and LV, turbulence was produced by passing linearly stratified steady flow through a square-bar mesh at the entrance to the working section. The turbulent kinetic energy equation (9) takes the form

\[ -\bar{u}_j \frac{\partial}{\partial x_j} \left( \frac{u'_i u'_j}{2} \right) = \frac{g}{\rho_0} \bar{w}' \rho' + \epsilon \]

\[ m_d = b + \epsilon \]  

(17)

Thus, \( R_f \) represents the efficiency with which the turbulent kinetic energy advected by the mean flow \( (m_d) \) is converted to the buoyancy flux term \( b \).

In the case of RIHV, turbulence was additionally produced downstream of the grid by a shear in the mean velocity profile, and the turbulent kinetic energy equation takes the form

\[ -\bar{u}_j \frac{\partial}{\partial x_j} \left( \frac{u'_i u'_j}{2} \right) - \bar{u}_j \frac{\partial}{\partial x_j} \bar{u}_i = \frac{g}{\rho_0} \bar{w}' \rho' + \epsilon \]

\[ m_d + m_s = b + \epsilon. \]  

(18)

Thus, \( R_f \) represents the efficiency with which the turbulent kinetic energy both advected by the mean flow \( (m_d) \) and generated locally by the shear production term \( (m_s) \) is converted to the buoyancy flux \( b \). In all cases the definition in (10) can be used to compute \( R_f \) from the experimental results tabulated in SHV, IHV, LV and RIHV.

a. Prandtl \( Pr > 1 \)

Consider first the salt-stratified water tunnel experiments where the Prandtl number was about 700. In Fig. 2 we show the data from these experiments plotted on an \( Fr_T - Re_T \) diagram where the data tabulated in the original papers were used and \( Fr_T \) and \( Re_T \) were computed from the definitions in (3) and (5). In general, the turbulent buoyancy flux will be nonzero provided there exists a range of overturning scales between the Ozmidov scale, where buoyancy affects the motion, and the Kolmogorov scale, where viscosity suppresses the motion (Gibson 1980). That is

\[ \alpha L_R > L_C > \beta L_K, \]

where \( \alpha \) and \( \beta \) are undetermined constants.

Buoyancy will first affect the turbulence when \( Fr_T = (L_R / L_C)^{2/3} = (\alpha)^{-2/3} \). Similarly, viscosity will suppress the turbulent motion when \( Re_T = (L_C / L_K)^{4/3} = (\beta)^{4/3} \), and the minimum requirement for overturning motion is
\[ \alpha L_R > \beta L_K \]

i.e.,

\[ \left( \frac{\epsilon}{\nu N^2} \right) = \text{Fr}_T^2 > \left( \frac{\beta}{\alpha} \right)^{4/3}. \tag{19} \]

The experiments by SHV, IHV and RIHV used direct measurements of the dissipation rate and displacement scale to evaluate both constants \( \alpha \) and \( \beta \). In contrast, Ivey and Nokes (1989) reported results from a very different experiment to the water tunnel experiments where mixing was driven by the breaking of critical internal waves on a sloping bottom in a uniformly stratified fluid. They did not measure dissipation rates directly, but rather computed the average dissipation in the turbulent region by measuring the work input to their closed system with a sensitive force transducer and accounted for losses to the observed increase in potential energy of the system and for laminar viscous dissipation (see Ivey and Nokes 1989 for details). While individual values of \( \alpha \) and \( \beta \) could not be found, the ratio of the two quantities was estimated and the results from these various experiments are compared in Table 1.

The data suggest that buoyancy first starts to affect the flow when \( \text{Fr}_T \approx 1 \), and viscosity suppresses the turbulence when \( \text{Re}_T \approx 15 \). In the experiment described by Ivey and Nokes (1989), while the transition to turbulence occurred at a smaller value of \( \text{Fr}_T \) than the other experiments, the four results in the last column of Table 1 indicate that overturning scales are suppressed by the combined action of buoyancy and viscosity when the value of \( \text{Fr}_T \approx (15)^{1/2} = 3.9 \). These three transitions are shown as the lines OA, OB and OC, respectively, in Fig. 2.

In order to test the prediction in (15) the value of the correlation coefficient \( R_{\rho w} \) must be known. In Fig. 3a, \( R_{\rho w} \) is plotted as a function of \( \text{Fr}_T \) for the experimental runs of SHV, IHV and RIHV. Figure 3a indicates a distinct change in the behavior of \( R_{\rho w} \) for \( \text{Fr}_T \) in the range 1–1.5. SHV, IHV and RIHV compared

<table>
<thead>
<tr>
<th>Source</th>
<th>( \alpha )</th>
<th>( \beta )</th>
<th>( \text{Fr}_T )</th>
<th>( \text{Re}_T )</th>
<th>( \frac{\text{Fr}_T^2}{\text{Re}_T} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>SHV</td>
<td>0.7</td>
<td>5.34</td>
<td>1.27</td>
<td>9.3</td>
<td>15</td>
</tr>
<tr>
<td>IHV</td>
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<td>8.38</td>
<td>1.11</td>
<td>17.0</td>
<td>21</td>
</tr>
<tr>
<td>RO</td>
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<td>8.8</td>
<td>0.94</td>
<td>18.2</td>
<td>16</td>
</tr>
<tr>
<td>Ivey and Nokes</td>
<td>1.11</td>
<td>14.8</td>
<td></td>
<td></td>
<td>15</td>
</tr>
</tbody>
</table>

Table 1. Summary of laboratory measurements.
the observed streamwise evolution of the displacement scale \( L_c \) for stable stratifications to the evolution of the overturning scale for a passive scalar, as observed by Montgomery (1974), to conclude that buoyancy first started to affect the flow when \( Fr_T \approx 1 \). Recalling that \( R_{sw} \) is independent of \( L_c \) by definition, the change in the character of \( R_{sw} \) around \( Fr_T \approx 1 \) is entirely consistent with the arguments of SHV, IHV and RIHV based on independent measurements in the same dataset.

Consider first the case when \( Fr_T > 1 \). Figure 3a indicates that beyond values of \( Fr_T \approx 1.0-1.5 \), \( R_{sw} \) tends to an asymptotic value in the range 0.2 to 0.5 and a representative value of \( (R_{sw})^{-1} \approx 3 \). The constancy of the correlation coefficient for \( Fr_T > 1 \) implies that the energetic turbulence has a well-developed velocity spectrum with the main contributions to the correlation coefficient coming from scales well removed from \( L_c \)—in fact the peak contribution comes from scales around \( 10 L_K \) (Itsweire and Helland 1989). This magnitude of \( (R_{sw})^{-1} \) implies that the second term in the denominator of (15) dominates and, as \( Fr_T \) increases to infinity, \( R_f \) tends to zero. In Fig. 4 we plot the value of \( R_f \) as a function of \( Fr_T \). The data indicate that at \( Fr_T \approx 1 \), \( R_f \) has a maximum of \( R_f \approx 0.20 \), and for increasing \( Fr_T \) the value of \( R_f \) decreases rapidly. Taking \( (R_{sw})^{-1} = 3 \) from Fig. 3a, we show in Fig. 4 the prediction from (15):

\[
R_f = \frac{1}{1 + 3Fr_T^2}, \quad Fr_T > 1.2. \tag{20}
\]

It is clear from Fig. 4 that this simple formulation is an excellent fit to the data covering the high \( Fr_T \) range for cases without a mean shear (SHV, IHV) or with a mean shear (RIHV). Note also, by reference to Fig. 2, that the data in Fig. 4 cover a large range of \( Re_T \) and the implication is the expression in (20) is indeed independent of \( Re_T \).

Consider now the case when \( Fr_T < 1 \). Figure 3a indicates the correlation coefficient \( R_{sw} \) decreases rapidly with decreasing \( Fr_T \), becoming zero at the line where

\[
Fr_T = 1.2.
\]
Fr_γ = 3.9 where the motion is made up predominantly of internal waves and the correlation between w' and p' is zero. Thus, the second term in the denominator of (15) will again dominate, only now through the influence of the rapid decrease in R_{pw}, and hence R_f approaches 0 as Fr_T decreases below 1. Note that these trends in R_{pw} imply that R_f is a maximum when Fr_T ~ 1.

For the range Fr_T < 1, we look for a functional form that describes the variation of R_f with Fr_T that satisfies three constraints. First, it must satisfy the condition that when Fr_T = 3.9[1/Re_T]^{1/2}, then R_f = 0. The functional form should also patch smoothly with the prediction in (20) at Fr_T = 1.2. Accordingly, we require at Fr_T = 1.2 that R_f = 0.19 and also dR_f/dFr_T = −0.25. These three constraints are satisfied by the parabola

\[ R_f = a + bFr_T + c(Fr_T)^2, \quad Fr_T < 1.2, \quad (21) \]

where

\[ a = 0.49 + 1.44c, \quad b = -0.25 - 2.4c, \]

\[ c = \frac{0.25\alpha - 0.49}{\alpha^2 - 2.4\alpha + 1.44}, \quad \alpha = 3.9 \left[ \frac{1}{Re_T} \right]^{1/2}. \]

In Fig. 5 we plot the observed value of R_f as a function of Fr_T in this regime. There is considerable scatter in the data, although it should be emphasized that, compared to the large range of Fr_T in Fig. 4, the data in Fig. 5 cover only a very small range of Fr_T (0.6 to 1.0) and is confined to the low Re_T part of the parameter space (see Fig. 2). For comparison we also plot in Fig. 5 the predicted efficiencies from (21) for Re_T = 40 (in the middle of the range of Re_T for the data points). There is some suggestion that the efficiency does decrease with decreasing Fr_T, although clearly data over a large range of Fr_T and hence large Re_T (see Fig. 2) is required to properly test the parameterization in (21) and refine this result. In Fig. 6a we plot all the data from the experiments, along with the predictions from Eqs. (20) and (21) (with Re_T = 40 in the midrange of the data in Fig. 2 for the latter case). This plot highlights the narrow range around Fr_T = 1 where R_f > 0.15.

Along the line OA in Fig. 2, the mixing efficiency is zero. In the experiments of Ivey and Nokes (1989), the flow was turbulent in a bottom boundary layer over the scale on which the mean shear was acting. In this case L_C ~ a, u ~ Na where a is internal wave amplitude, and thus from (3)

\[ Fr_T = \frac{u}{NL_C} = \left( \frac{\Gamma}{N} \right) \sim 1. \]

If Fr_T ~ 1, we see for this special case from (8) that \( Re_T \sim (Fr_T)^2 = \epsilon/(\nu N^2) \). The mixing efficiencies measured by Ivey and Nokes (1989) are therefore plotted in Fig. 7 as a function of \( (\epsilon/\nu N^2) \). For small values of Re_T, no change in the potential energy of the stratification was observed and hence R_f = 0. Above a critical Reynolds number Re_TC of about 8, the mixing efficiency increased until, for Re_T > 2 Re_TC, the mixing efficiency approached an asymptotic value of about 0.20—the same upper bound observed in the water tunnel experiments of SHV, IHV and RIHV. The reason for the similar value is that this particular mechanism is operating near the maximal efficiency expected for any turbulent process with Fr_T ~ 1. Since the precise value of Fr_T was not measured in the experiments of Ivey and Nokes (1989), it is not possible to make exact comparisons with the other laboratory data summarized in Fig. 2. In the context of Fig. 2, however, the implication is that as the Reynolds number decreases below Re_T ~ 30 (twice the transition value found in Table 1), the mixing efficiency should decrease below the prediction in (20) and drop to zero along line OA with Re_T = 15.

On Fig. 2 we plot the curve for a mixing efficiency of R_f = 0.15, as given by (20) and (21), and this emphasizes how most of the domain in the Fr_T–Re_T diagram has a mixing efficiency R_f < 0.15.

b. Prandtl Pr < 1

Let us now consider the experiments of Lienhard and Van Atta (1990) (LV) carried out in a wind tunnel where the incoming air is electrically heated to produce a vertical temperature gradient in air with a Prandtl of 0.7. The correlation coefficient R_{pw} is plotted as a function of Fr_T in Fig. 3b. The figure shows the correlation coefficient is zero for small Fr_T increases to a maximum at Fr_T ~ 2, and then remains approximately constant as Fr_T is increased further. The general behavior is identical to that of the large Prandtl number case shown in Fig. 3a.

While the general trends are the same in Figs. 3a and 3b, there are two differences between the low and high Prandtl cases shown. First, the peak in the correlation coefficient is reached at Fr_T ~ 1.5–2.0 for the case of Pr = 0.7, compared to a peak at Fr_T ~ 1.0–1.5 for the case of Pr = 700. This will have the effect of slightly shifting the peak in the mixing efficiency.

![Fig. 5. Flux Richardson number R_f as a function of Fr_T for the data in Fig. 1 for points with Fr_T < 1. The curve is the predicted fit in Eq. (21).](image-url)
curve. Second, for large values of \( \text{Fr}_T \) the correlation coefficient is approximately 0.6 for the case of \( \text{Pr} = 0.7 \), compared to 0.3 for the case of \( \text{Pr} = 700 \).

As Lienhard and Van Atta (1990) argue, variations of \( \rho' \) are rapidly dissipated in the air flows with \( \text{Pr} = 0.7 \), persisting only through ongoing turbulent mixing and thus \( w' \) and \( \rho' \) must be strongly correlated for \( \text{Pr} < 1 \). All contributions to \( w' \rho' \) must come from fluctuations in both \( w' \) and \( \rho' \) with scales larger than the Kolmogorov scale. For example, Itsweire and Helland (1990) show for the salt stratified experiments that the smallest scale contributing to the density flux is about 3 times the Kolmogorov scale, and the peak contribution comes from scales around 10 times the Kolmogorov scale. In the case \( \text{Pr} < 1 \), fluctuations in \( \rho' \) are directly forced at small scales by the velocity fluctuations and diffusion annihilates fluctuations in density, which are at scales smaller than the velocity scales. This implies a close correlation between density and velocity over the whole range of contributing length scales and thus a high correlation coefficient. In the case \( \text{Pr} > 1 \), density perturbations are dissipated much less rapidly than the velocity fluctuations and can persist long after new velocity scales are formed at the Kolmogorov scale. Thus a smaller correlation coefficient is to be expected for the case \( \text{Pr} > 1 \).

Apart from these differences, the behavior appears the same in the two cases and in Fig. 6b we show both the data from LV and the model predictions for mixing efficiency. The model curves are
a constant fraction of the surface buoyancy flux and, consistent with scaling arguments (Imberger 1985; Imberger and Ivey 1991) is given by
\[ \epsilon = -0.45 b_0. \] (25)

Substituting (24) and (25) into (10) yields
\[ R_f = \frac{b}{m} = \frac{1}{\left(1 - \frac{0.45}{1 - 1.1(z/h)}\right)}. \] (26)

Equation (26) exhibits a simple depth dependence of \( R_f \). Since the buoyancy flux term \( b \) is sustaining the turbulence, it is perhaps most appropriate to think of the variation of \( R_f^{-1} \) with depth. Note also that the results in section 4 imply there may be a weak Prandtl number dependence of the coefficients in (24) and (25). In any case, (26) clearly shows that at the surface \( z = 0, R_f = 2.18 \) (or \( R_f^{-1} = 0.55 \)), and the introduced buoyancy flux \( b_0(<0) \) is partitioned between turbulent dissipation \( \epsilon (>0) \) and mechanical turbulent kinetic energy flux \( m(<0) \), which is then exported downwards. At \( (z/h) = 0.5 \), the sign of the mechanical flux term changes \( (m = 0) \) and \( R_f = \alpha \) (or \( R_f^{-1} = 0 \)). For greater depths the excess mechanical flux \( m(>0) \) is used to transport heat up from the base of the surface layer against the stable mean temperature gradient in this region until at \( (z/h) = 0.9 \) the buoyancy flux \( b = 0 \) and \( R_f = 0 \) (or \( R_f^{-1} = -\alpha \)) (see Imberger and Ivey 1991 for a detailed discussion).

5. Discussion

The preceding results indicate that, providing the sign of the buoyancy flux is known, the location of a particular mixing event in \( \text{Fr}_T - \text{Re}_T \) space determines the magnitude of the flux Richardson number \( R_f \). The location of a mixing event in \( \text{Fr}_T - \text{Re}_T \) space is determined by the type of mechanism. For example, Imberger and Ivey (1991) note that Kelvin-Helmholtz billows will have \( \text{Fr}_T \approx 1 \), and hence the prediction from the above is that the mixing efficiency for this particular mechanism should be around 0.20—independent of \( \text{Re}_T \) provided \( \text{Re}_T > 30 \).

Garrett and Gilbert (1988) have theoretically investigated the mixing due to the breaking of internal waves on sloping boundaries in the deep ocean, and their calculations require an estimate of mixing efficiency. The \( \text{Re}_T \) in the field will be considerably larger than the modest \( \text{Re}_T \) number obtainable by Ivey and Nokes (1989) in their laboratory simulations of the same mechanism. As the observations of Ivey and Nokes (1989) showed, however, the mixing for critical internal wave breaking is one where the mixing is driven by a shear adjacent to the boundary where the scale of the overturning was on the scale of the shear and, thus, just as in the case of Kelvin-Helmholtz billows, \( \text{Fr}_T \approx 1 \) (see Imberger and Ivey 1990). For this case with \( b > 0 \), the implication is that while \( \text{Re}_T \) is much greater in the field the mixing efficiency should be the same as the maximum value of about 0.20 observed in the laboratory experiments.
It is important to note that the preceding conclusions are generally valid for fluids with a Prandtl number equal to 0.7 or 700. While there are small differences brought about by the higher correlation coefficient for the case Pr = 0.7, the trends in the values of the flux Richardson number remain the same in both cases (see Figs. 6a and 6b). Imberger and Ivey (1991) mostly discuss data for thermally stratified water (Pr = 7). For this case, the Batchelor scale is smaller than the Kolmogorov scale, just as in the case of the salt stratified experiments, and so we may expect the efficiencies to be given by (20) and (21). The temperature and salinity fluctuations will only differ at scales smaller than the thermal Batchelor scale, and these do not contribute to the correlation between w′ and ρ′.

Preliminary field data derived from a profiling microstructure flux probe (MFP) in a shear driven thermally stratified reservoir indicate this to be true. This untethered instrument rises through the water column, measuring at 1 mm resolution u′ and w′ with a laser Doppler system, θ′ with a fast response thermistor, and s′ (salinity) with a miniature four-electrode conductivity probe. The correlations derived from three segments collected in a local water supply reservoir are shown in Fig. 3a and the measured efficiencies are included in Fig. 6a. While preliminary, the data provide excellent confirmation for the above interpretation and lend confidence to the ability to transfer inferences from laboratory data to the field.

6. Conclusions

Using scaling arguments and laboratory observations, we have shown that the magnitude of the flux Richardson number Rf is quantitatively determined by the location of the event in the Ff-Pr-Ref diagram.

These results indicate that for mechanically energized turbulence (m > 0), the value of Rf varies between 0 and 0.02 for a fluid with Prandtl number greater than 1, and between 0 and 0.15 for a fluid with a Prandtl number less than 1. Comparisons with preliminary results from field measurements in shear-driven thermally stratified water are in good agreement with the predictions.

For turbulence energized by a negative buoyancy flux, Rf−1 is the more relevant measure of the energetics of the mixing and the value of Rf−1 varies with depth between 0.54 at the surface towards −∞ near the base of the surface layer.

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