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4. THE COASTAL JET CONCEPTUAL MODEL IN THE DYNAMICS OF SHALLOW SEAS

G. T. Csanady

1. Introduction

In the atmospheric sciences we deal with fluid motions of great complexity depending on a very large number of external and internal parameters. Moreover, in most problems we are interested in enough of the details of those motions so that their statistics alone do not satisfy us. Therefore the two usual approaches of physical theory become inappropriate: we cannot isolate a limited number of independent parameters which fully determine a phenomenon of interest (in the manner of a controlled laboratory experiment), nor is it fruitful to consider many realizations of the phenomenon concerned and confine attention to its average properties alone (in the manner of the kinetic theory of gases). One response to this challenge that has emerged in the atmospheric sciences is a theoretical tool that may be described as the "conceptual model." As we shall understand it, such a model is deliberately incomplete and aims only at reproducing predominant characteristics of some distinctly identifiable, observable phenomenon. One good example of a "distinct" phenomenon in oceanography is the Gulf Stream, noting at once that its distinctness is clear enough in a gross way, but a precise definition of its boundaries would be problematic. It is also clear that the Stream interacts in a complex way with its environment, and yet there is no question of the usefulness of regarding it as a separate entity. An example of the conceptual model is Stommel's (1948) theory of westward intensification, which has allowed us a great deal of insight into the dynamics of the Stream. The key steps in gaining this insight have been: (a) the distillation from complex experience of a clearly identifiable, distinct flow structure called the Gulf Stream; and (b) the construction of a model ocean of very simple properties, but containing just enough complexity to produce a western boundary current of gross characteristics quantitatively very similar to the Gulf Stream. It is likely that in the atmospheric sciences dynamical understanding of complex phenomena will forever be based on such incomplete conceptual models, even if our models become more and more sophisticated as a given branch of the science advances. This chapter is devoted to the discussion of one such conceptual model which has provided insight into the dynamics of nearshore water movements in shallow seas.

To be more specific, the tidal equations have some aperiodic solutions that provide simple but realistic models of certain wind-driven flow problems. The first such model was proposed by Charney (1955) in a discussion of geostrophic adjustment near a coastline, and was named by him the "coastal jet." Although intended originally to elucidate Gulf Stream dynamics, the model has proved useful in connection with directly (locally) wind-driven coastal currents (which the Gulf Stream is not) in the Great Lakes and over continental shelves. A grossly simplified summary of the dynamical principles involved is as follows. We understand wind-driven flow in the deep ocean in terms of the conceptual model known as "Ekman drift," in which the Coriolis force associated with crosswind water transport balances the stress exerted by the wind over the water surface. A coastline parallel to the wind, however, prevents Ekman drift. In a conceptual model without bottom or side friction, the wind stress then accelerates the water within some nearshore region and produces a "coastal
jet." In a basin of constant depth the width of the jet turns out to be proportional to the Rossby radius of deformation. Especially relevant to observation are internal mode (baroclinic) coastal jets, which simulate clearly identifiable wind-driven coastal currents.

As may be expected, various complications arise due to the finite length and width of basins (lakes or continental shelves) and to their variable depths. In the following sections we first give an account of the coastal jet conceptual model, beginning with the crudest approach and making it gradually more realistic, and then illustrate some observed coastal currents which the coastal jet model is intended to simulate.

2. The Simplest Coastal Jet Model: Semi-infinite Ocean

Consider a semi-infinite ocean bounded by a straight coastline coincident with the \( y \) axis, with the \( x \) axis pointing out to sea. Assume that the water depth is a constant \( H = h + h' \), a slightly lighter top layer of equilibrium depth \( h \) lying over dense water of depth \( h' \). The fractional density defect \( \epsilon = (\rho' - \rho)/\rho \) is small, of order \( 10^{-3} \). Let a wind begin to blow at time \( t = 0 \), exerting a stress \( \tau_0 = \rho F \) at the water surface, directed parallel to the coast, constant in space and in time for \( t > 0 \).

For a sufficiently short period the motions generated in the water will be slow enough for nonlinear accelerations to be neglected. Also, friction at the stable interface and at the bottom will for a time be unimportant, as will frictional stresses in vertical planes. Therefore we can elucidate some important properties of the initial motion with the aid of the linearized equations of momentum and mass balance integrated over depth separately for top and bottom layers. The pressure is assumed to be hydrostatic, and the pressure gradients expressed in terms of surface and thermocline elevations above equilibrium, \( \zeta \) and \( \zeta' \), both supposed small compared to equilibrium depths. The resulting well-known transport equations are as follows:

Top layer:

\[
\frac{\partial U}{\partial t} - fV = -gh \frac{\partial \zeta}{\partial x} \\
\frac{\partial V}{\partial t} + fU = -gh \frac{\partial \zeta}{\partial y} + F \\
\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = -\frac{\partial}{\partial t}(\zeta - \zeta')
\]

Bottom layer:

\[
\frac{\partial U'}{\partial t} - fV' = -gh' \frac{\partial \zeta}{\partial x} + gh'e \frac{\partial}{\partial x} (\zeta - \zeta') \\
\frac{\partial V'}{\partial t} + fU' = -gh' \frac{\partial \zeta}{\partial y} + gh'e \frac{\partial}{\partial y} (\zeta - \zeta') \\
\frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = -\frac{\partial \zeta'}{\partial t}
\]

(1)

where \( U, V, U', \) and \( V' \) are depth-integrated velocities or volume transports (separately for top and bottom layers) along \( x \) and \( y \).
Boundary conditions at the coast \( x = 0 \) are such that the normal transports \( U \) and \( U' \) vanish. At infinity, \( x \to \infty \), we postulate vanishing surface and interface elevations.

The above sets of equations for top and bottom layers are clearly coupled through the presence of both \( \zeta \) and \( \zeta' \) in either set. It is possible, however, to produce two linear combinations of the two sets in such a way that the resulting sets of equations are uncoupled normal mode equations of the form:

\[
\frac{\partial U_n}{\partial t} - f V_n = -g h_n \frac{\partial \zeta_n}{\partial x}
\]
\[
\frac{\partial V_n}{\partial t} + f U_n = -g h_n \frac{\partial \zeta_n}{\partial y} + F_n
\]
(2)
\[
\frac{\partial U_n}{\partial x} + \frac{\partial V_n}{\partial y} = -\frac{\partial \zeta_n}{\partial t}
\]

where \( h_n \) is an appropriate “equivalent depth,” although it may also be regarded a factor modifying gravity. These equations describe motions of a homogeneous fluid of depth \( h_n \). Equations 2 arise from adding a constant \( \alpha_n \) times the first set of three equations of equation 1 to the second set of three, so that the combined variables are

\[
U_n = \alpha_n U + U'
\]
\[
V_n = \alpha_n V + V'
\]
\[
\zeta_n = \alpha_n(\zeta - \zeta') + \zeta'
\]
\[
F_n = \alpha_n F
\]
(3)

These linear transformations, when introduced into equations 1 result in a set of equations of the form of equation 2, provided that \( \alpha_n (n = 1, 2) \) is a root of Stokes's equation:

\[
\alpha^2 + \left( \frac{h'}{h} - 1 \right) \alpha - \frac{h'}{h} (1 - \varepsilon) = 0
\]
(4)

The corresponding equivalent depths are then

\[
h_n = h' \varepsilon (1 - \alpha_n)^{-1}
\]
(5)

The above separation of the problem into normal modes is exact under the postulated conditions. The physical properties of the normal mode equations are best exhibited, however, using approximate values of the roots \( \alpha_{1, 2} \), valid for small values of the density defect \( \varepsilon \). The two roots are, expanded in powers of \( \varepsilon \), as follows:

\[
\alpha_1 = 1 - \frac{\varepsilon h'}{h + h'} + 0(\varepsilon^2)
\]
\[
\alpha_2 = -\frac{h'}{h} + \frac{\varepsilon h'}{h + h'} + 0(\varepsilon^2)
\]
(6)
with corresponding equivalent depths, to the lowest order only:

\[ h_1 = h' + h + 0(\varepsilon) \]

\[ h_2 = \frac{\gamma h'}{h + h'} + 0(\varepsilon^2) \]  

To lowest order in \( \varepsilon \), the normal mode equations (to be called “surface” or “barotropic” and “internal” or “baroclinic” modes) are now written in terms of the original variables, according to equation 3, but multiplied by a constant factor in the case of the internal mode:

Surface mode:

\[ \frac{\partial(U + U')}{\partial t} - f(V + V') = -g(h + h') \frac{\partial \zeta}{\partial x} \]

\[ \frac{\partial(V + V')}{\partial t} + f(U + U') = -g(h + h') \frac{\partial \zeta}{\partial y} + F \]

\[ \frac{\partial(U + U')}{\partial x} + \frac{\partial(V + V')}{\partial y} = - \frac{\partial \zeta}{\partial t} \]  

Internal mode:

\[ \frac{\partial U'}{\partial t} - f V' = -g \epsilon \frac{hh'}{h + h'} \frac{\partial \zeta'}{\partial x} \]

\[ \frac{\partial V'}{\partial t} + f U' = -g \epsilon \frac{hh'}{h + h'} \frac{\partial \zeta'}{\partial y} - \frac{h'}{h + h'} F \]

\[ \frac{\partial U'}{\partial x} + \frac{\partial V'}{\partial y} = - \frac{\partial \zeta'}{\partial t} \]  

The approximate surface mode equations are exactly those that apply to a homogeneous fluid of depth \( h + h' \), acted upon by a wind stress \( \rho F \), along the \( y \) axis. In this mode (and to zeroth order) the fluid does not “feel” the small density defect of the top layer. The internal mode equations are identical in form to the surface mode ones, when applied to motions of the bottom layer of fluid alone, that is, as if the thermocline were a free surface, but with gravity reduced by a factor of \( \gamma h/(h + h') \). Also the effective force acting on the bottom layer is opposite in direction to the applied wind stress, and its magnitude is reduced by the factor \( h'/(h + h') \). The reduction in gravity is by a large factor, whereas the effective stress is not much different from that acting at the surface. One may therefore at once suspect that very large thermocline displacements may be produced. The opposite direction of the effective force in the internal mode has the consequence of producing opposite surface and thermocline displacements.

The full solution of the problem we posed (suddenly imposed wind) contains some inertial oscillations and an aperiodic part describing a “coastal jet” structure near shore and Ekman drift far offshore. Full details are given by Crépon (1967); here we
concern ourselves only with the aperiodic part of the response. Solving the approximate normal mode equations 8 separately, and adding the results, we arrive at the particular solution (correct to order \( \epsilon^{1/2} \)):

\[
\begin{align*}
    u &= \frac{U}{h} = \frac{F}{f(h + h')} \left\{ 1 - \exp \left( -\frac{x}{R_1} \right) + \frac{h'}{h} \left[ 1 - \exp \left( -\frac{x}{R_2} \right) \right] \right\} \\
    v &= \frac{V}{h} = \frac{F t}{h + h'} \left[ \exp \left( -\frac{x}{R_1} \right) + \frac{h'}{h} \exp \left( -\frac{x}{R_2} \right) \right] \\
    \zeta &= -\frac{F t}{f R_1} \left[ \exp \left( -\frac{x}{R_1} \right) + \frac{h' R_2}{h R_1} \exp \left( -\frac{x}{R_2} \right) \right] \\
    u' &= \frac{U'}{h'} = \frac{F}{f(h + h')} \left\{ 1 - \exp \left( -\frac{x}{R_1} \right) - \left[ 1 - \exp \left( -\frac{x}{R_2} \right) \right] \right\} \\
    v' &= \frac{V'}{h'} = \frac{F t}{h + h'} \left[ \exp \left( -\frac{x}{R_1} \right) - \exp \left( -\frac{x}{R_2} \right) \right] \\
    \zeta' &= \frac{F t}{f R_2} \frac{h'}{h + h'} \left[ \exp \left( -\frac{x}{R_2} \right) - \frac{R_2}{R_1} \exp \left( -\frac{x}{R_1} \right) \right]
\end{align*}
\]

(9)

Here \( u, v, \text{etc.} \) are velocities averaged over the two layers separately and the distances \( R_{1,2} \) are surface and internal radii of deformation:

\[
\begin{align*}
    R_1 &= f^{-1} \sqrt{g(h + h')} \\
    R_2 &= f^{-1} \sqrt{g e h h'/(h + h')}
\end{align*}
\]

(10)

These distances give the widths of coastal jets: as equations 9 show, within bands of scale width \( R_1 \) and \( R_2 \) longshore velocities, surface, and thermocline elevations increase in direct proportion to time or, more descriptively, to the impulse \( F t \) of the wind stress. The onshore–offshore velocity reduces to zero on approaching the coast within a band of scale width \( R_1 \) and \( R_2 \), referring to the contribution of the surface and internal mode, respectively.

Very close to shore, \( x \ll R_2 \) (note that \( R_2 \ll R_1 \), by equation 10) the onshore velocities \( u \) and \( u' \) in either layer are negligible. The longshore velocities are, on the other hand, to a high degree of approximation:

\[
\begin{align*}
    v &= \frac{F t}{h} \quad (x \ll R_2) \\
    v' &= 0
\end{align*}
\]

(11)

In other words, the impulse of the wind stress is distributed over the top layer only. The physical reason is contained in the fifth of equations 1: in the absence of pressure gradients along \( y \) (and of friction at the interface), bottom layer longshore velocity can be generated only by Coriolis force due to onshore–offshore flow. Because of the shore constraint this is zero; hence the water of the bottom layer must remain stagnant.
At intermediate values of \( x \), \( R_2 \ll x \ll R_1 \), onshore–offshore velocities are approximately

\[
\begin{align*}
    u &= \frac{F}{f(h + h')} \frac{h'}{h} \\
    u' &= -\frac{F}{f(h + h')} \\
(12) \quad (R_2 \ll x \ll R_1)
\end{align*}
\]

The total offshore transport is thus zero, \( uh + u'h' = 0 \), outflow in the top layer (if \( F \) is positive) being balanced by inflow in the bottom layer. Longshore velocities are in this same intermediate region:

\[
    v = \frac{Ft}{h + h'} = v' \quad (R_2 \ll x \ll R_1)
\]  
(13)

The impulse of the wind stress has been distributed evenly here over the entire water column. Within the bottom layer this has happened through the medium of the Coriolis force, because by the fifth of equations 1,

\[
    V' = -\int_0^t f \, U' \, dt = \frac{Ft}{h + h'} h'
\]  
(14)

bottom layer onshore transport having been substituted from equation 12. The total onshore displacement in the bottom layer is thus just sufficient to produce the required longshore transport through the action of the Coriolis force.

Far from shore, \( x \ll R_1 \), longshore velocities are zero, the bottom layer is completely at rest, and Ekman drift in the top layer balances wind stress.

Turning now to surface and thermocline elevations, we see that these increase in proportion to time, or rather to wind-stress impulse, and are present at any distance not large compared to \( R_1 \). Because \( R_2 \ll R_1 \), thermocline elevations are large, but only close to shore, \( x \ll R_2 \). It is easily shown that the total accumulation of bottom layer water near shore, beneath the rapidly rising part of the thermocline, is exactly what is flowing inshore in the “intermediate” region (equation 12). It is also easily checked that this water is transferred from a region of scale width \( R_1 \) (where the thermocline is being slowly depressed) to an interior region of scale width \( R_2 \) (where it is rapidly rising). Corresponding to the much larger extension of the \( R_1 \) region, the depression of the thermocline at any instant is small compared to its elevation over the \( R_2 \) region.

The surface elevation is depressed (by a positive \( F \)) over both \( R_1 \) and \( R_2 \) regions, to the total extent required by the Ekman drift far offshore. At the shore, however, the depression is greater than if the fluid were homogeneous. Even if \( R_2/R_1 \) is small, this extra elevation may be detectable, because the combined factor \( h'R_2/hR_1 \) may not be much smaller than unity.

We also note that, given time-independent onshore–offshore flow, the Coriolis force of longshore flow at any instant is balanced by onshore–offshore pressure gradients. The momentum balance along \( x \) is thus geostrophic, with both longshore flow and offshore pressure gradient increasing linearly in time.

The above results are best appreciated if one considers “typical” magnitudes of the quantities involved. Table I lists plausibly assumed values of the independent parameters characterizing shallow seas. The ratio of the radii of deformation is seen to be typically about 50, and the magnitude of \( R_1 \) is so great that \( x \gg R_1 \) is not a
realistic part of the model: no shallow sea is more than 1000 km wide. However, the nearshore and intermediate regions may be realistically described by this model. The validity of our results must be restricted to an initial period. We make calculations for $t = 2.10^4$ sec, or just under 6 hr.

Equations 9 show that the velocities are scaled by $F/f(h + h')$ and $Fr/(h + h')$: these become 1 and 2 cm/sec, respectively, with the above assumptions. Surface elevations are proportional to $Fr/R_1 = 6.32$ cm, thermocline elevations to $[Fr/R_2][h/(h + h')] = 283$ cm. The most conspicuous effect of the wind is clearly the movement of the thermocline, which rapidly outgrows the linearizing assumption of $|\zeta| \ll h$. Calculated characteristics are listed in Table II. The coastal jet is "baroclinic" near shore (confined to the top layer, and associated with an inclined thermocline), and "barotropic" in the intermediate region (evenly distributed over the depth, and associated with a surface slope). Figure 1 is an attempt to illustrate the properties of a two-layer coastal jet structure with a wind tending to produce upwelling. The opposite wind produces an opposite jet and a downwelled thermocline, of otherwise the same characteristics.

---

**Table I**

"Typical" Data of Shallow Seas

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Total depth $h + h'$</td>
<td>$10^4$ cm</td>
</tr>
<tr>
<td>Top layer $h$</td>
<td>$2.10^3$ cm</td>
</tr>
<tr>
<td>Wind force $\tau_0/\rho = F$</td>
<td>$1$ cm$^2$/sec$^2$</td>
</tr>
<tr>
<td>Surface mode radius of deformation $R_1$</td>
<td>$316$ km</td>
</tr>
<tr>
<td>Bottom layer $h'$</td>
<td>$8.10^3$ cm</td>
</tr>
<tr>
<td>Effective gravity $eg$</td>
<td>$2$ cm/sec$^2$</td>
</tr>
<tr>
<td>Internal mode radius of deformation $R_2$</td>
<td>$5.66$ km</td>
</tr>
<tr>
<td>Coriolis parameter $f$</td>
<td>$10^{-4}$/sec</td>
</tr>
</tbody>
</table>

---

Fig. 1. Schematic picture of coastal jet development in two-layer fluid.
TABLE II

Some Characteristics of Two-Layer Coastal Jet

<table>
<thead>
<tr>
<th>Variable</th>
<th>Close to Shore $x \ll R_2$</th>
<th>Intermediate $R_2 \ll x \ll R_1$</th>
<th>Far $x \gg R_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u$ (cm/sec)</td>
<td>0</td>
<td>4</td>
<td>5</td>
</tr>
<tr>
<td>$v$ (cm/sec)</td>
<td>10</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\zeta$ (cm)</td>
<td>$-6.46$</td>
<td>$-6.32$</td>
<td>0</td>
</tr>
<tr>
<td>$u'$ (cm/sec)</td>
<td>0</td>
<td>$-1$</td>
<td>0</td>
</tr>
<tr>
<td>$v'$ (cm/sec)</td>
<td>0</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>$\zeta'$ (cm)</td>
<td>$275$</td>
<td>$-5.06$</td>
<td>0</td>
</tr>
</tbody>
</table>

3. Opposing Shores

The most important inadequacy of the preceding section's simple model is that a region “far” from shore does not exist in reality, yet the boundary conditions imposed there play a certain role in determining the structure of the solution. The simplest “fix” for this problem is to assume a second parallel shore at $x = a$, that is, to use a canal model in place of a semi-infinite ocean.

The full solution of the initial value problem for the canal model is also available (Crépon, 1969) and is not very different from the previous case. If one takes the $y$ axis to be at the center of the canal, the aperiodic part of the solution becomes symmetrical or antisymmetrical, containing hyperbolic functions in place of the exponentials in equation 9. To order $\varepsilon^{1/2}$ the solution is

$$
\begin{align*}
    u &= \frac{F}{f(h + h') \left( 1 - \frac{\cosh(x/R_1)}{\cosh(a/2R_1)} \right)} + \frac{h'}{h} \left( 1 - \frac{\cosh(x/R_2)}{\cosh(a/2R_2)} \right), \\
    v &= \frac{Ft}{h + h'} \left( \frac{\cosh(x/R_1)}{\cosh(a/2R_1)} + \frac{h'}{h} \frac{\cosh(x/R_2)}{\cosh(a/2R_2)} \right), \\
    \zeta &= -\frac{Ft}{fR_1} \left( \frac{\sinh(x/R_1)}{\sinh(a/2R_1)} + \frac{h'}{h} \frac{R_2}{R_1} \frac{\sinh(x/R_2)}{\sinh(a/2R_2)} \right), \\
    u' &= \frac{F}{f(h + h') \left( 1 - \frac{\cosh(x/R_1)}{\cosh(a/2R_1)} \right)} - \left( 1 - \frac{\cosh(x/R_2)}{\cosh(a/2R_2)} \right), \\
    v' &= \frac{Ft}{h + h'} \left( \frac{\cosh(x/R_1)}{\cosh(a/2R_1)} - \frac{\cosh(x/R_2)}{\cosh(a/2R_2)} \right), \\
    \zeta' &= \frac{Ft}{fR_2(h + h')} \left( \frac{\sinh(x/R_1)}{\sinh(a/2R_1)} - \frac{R_2}{R_1} \frac{\sinh(x/R_2)}{\sinh(a/2R_2)} \right).
\end{align*}
$$

The case of interest is when $a \ll R_1$; in this case the “intermediate” region $R_2 < x < R_1$ occupies most of the canal, outside boundary layers of scale width $R_2$. At one shore the thermocline is rising; at the other it is sinking. Fluid in the top layer is transferred effectively from one shore to the other, to the right of the wind, while fluid in the bottom layer compensates. Over most of the canal the wind-imparted momentum is distributed evenly over the water column. In the bottom layer this is
brought about by the Coriolis force due to cross flow. The same effect in the top layer subtracts the appropriate amount of momentum from the wind input. The surface elevation distribution is nearly

$$\zeta = -\frac{Fr}{fR_1} \frac{x}{2a}$$  \hspace{1cm} (16)

the slight constant cross-canal slope balancing the Coriolis force of longshore flow.

Within the coastal boundary layers longshore flow is confined to the top layer and conditions are in every way exactly as in the case of the semi-infinite ocean. Figure 2 illustrates the canal model of coastal jets.

### 4. End Walls

Another unrealistic feature of the simple coastal jet model is the infinite extension of the coast along the $y$ axis. Closing the ends at $y = \pm b/2$ produces a more realistic model of a closed basin at the expense of some rather drastic modifications of the dynamical response. Although no part of the response of a closed basin is strictly linear in time, the coastal jet type of response emerges again as the initial behavior of low-frequency long waves, with some important additional features and qualifications.

The response of a rectangular, two-layer basin of constant depth to a suddenly imposed constant wind stress consists of a time-independent (static) particular solution and a series of free oscillations. Most of the free oscillations have the character of the well-known seiches (Wilson, 1972) or Poincaré waves (Mortimer, 1963), the frequencies of which are higher than inertial. From our point of view, these oscillatory solutions applying to a simplified basin constitute conceptual models fundamentally different from coastal jets; we are not concerned with them here. However, in sufficiently large basins some wavelike modes have a frequency much less than $f$ and
may be combined with the static solution to provide a quasi-static response pattern of the kind we are discussing here.

The solution is most conveniently discussed in connection with equations 2, that is, separately for the surface and internal modes. In either mode, \( n = 1 \) or 2, the static solution is

\[
\zeta_n = \frac{F_n}{f^2 R_n^2} \left( \sum_{j=1}^{\infty} (-1)^j \frac{4R_j^2/R_n^2 \sinh(y/R_j)}{(2j - 1) \sinh(b/2R_j)} \cos \left( (2j - 1) \frac{\pi x}{a} \right) \right)
\]

(17)

where

\[
\frac{1}{R_j^2} = \frac{1}{R_n^2} + \frac{(2j - 1)^2\pi^2}{a^2} \quad (j = 1, 2, 3, \ldots)
\]

(17a)

and \( R_n \) have the same definitions as in equation 10. The corresponding equivalent transport components are

\[
U_n = \frac{\partial \psi_n}{\partial y} \quad V_n = -\frac{\partial \psi_n}{\partial x}
\]

(18)

where

\[
\psi_n = f R_n^2 \zeta_n - \frac{F_n}{f} y
\]

(18a)

so that \( \psi_n \) is volume-transport stream function in the "equivalent" basin.

The properties of this solution depend on the size of the equivalent basin in comparison with the radius of deformation. For a small basin, \( a/R_n, b/R_n \to 0 \), the terms of the trigonometric series in the brackets are all negligible and the hyperbolic cosines are unity, so that, approximately

\[
\zeta_n = \frac{F_n}{f^2 R_n^2} y \quad \left( \frac{a}{R_n}, \frac{b}{R_n} \to 0 \right)
\]

(19)

Equation 18 now shows that the volume transports are zero to the same approximation. This solution is known as a static "setup," the surface slope of the water just balancing the wind stress everywhere.

In the large basin limit \( (a/R_n, b/R_n \to \infty) \) the hyperbolic functions in the denominators in equation 17 are very large, so that \( \zeta_n \) is nearly zero everywhere, except where at least some of the numerators are also large, that is, within distances of order \( R_n \) from boundaries. Along the side walls, \( x = \pm a/2 \), at least several \( R_n \) from the end walls, the elevation distribution is exactly as given by equation 19, but this drops to zero within a coastal boundary layer of scale width \( R_n \). At the center of the basin the elevation is zero, and the transport is Ekman transport balancing wind stress. At the end walls the elevations are \( \pm F_n b/2f^2 R_n^2 \), that is, also as in the small basin, but again the drop along a line parallel to the \( y \)-axis is not gradual but sudden and is accomplished within a distance of order \( R_n \). The equivalent volume transports \( U_n, V_n \) are as follows:

\[
U_n = \frac{F_n}{f}, \quad V_n = 0 \quad \text{(center portion)}
\]

\[
U_n = \pm \frac{b}{2R_n} \frac{F_n}{f}, \quad V_n = 0 \quad \text{(end walls)}
\]

(20)

\[
U_n = 0, \quad V_n = \pm \frac{y}{R_n} \frac{F_n}{f} \quad \text{(side walls)}
\]
This flow pattern is illustrated in Figure 3 for the specific case of $b/a = 5, a/R_n = 15$. In applying these results to realistically large basin models the important point is that the external radius of deformation $R_1$ is generally larger than either $a$ or $b$, while the internal radius $R_2$ is much smaller. Thus the surface mode contribution to surface elevation distribution is nearly, from equation 19:

$$\zeta = \frac{F}{g(h + h')} y$$  \hspace{1cm} (19a)$$

Corresponding to this result the depth-integrated total transports $U + U'$, $V + V'$ are very nearly zero everywhere, so that the balance of forces is between wind stress and pressure gradient, or static setup.

The opposite limit solution (basin size much larger than $R_n$) applies to the internal mode in sufficiently large basins. The surface layer motion in such basins becomes similar to the pattern shown in Figure 3. Bottom layer transports are equal and opposite, in order to bring the total transport to zero.

The above, we recall, was a steady-state particular solution. To construct a total solution now, satisfying the initial condition of a lake at rest, it is necessary to add a
series of wavelike components which at \( t = 0 \) just cancel the static solutions. In the surface mode this results in the addition of a number of seiches, mostly parallel to lake axes and having frequencies of order \( R_1 f/b \) and higher, which are by hypothesis large compared to \( f \). In the internal mode, on the other hand, the side wall boundary layer flows are canceled out by Kelvin waves, the amplitudes of which diminish to zero within a distance of order \( R_2 \) from the boundary, and the frequencies of which are, in the gravest modes, of order \( f R_2 / b_1 \), that is, much lower than \( f \). Figure 4 illustrates this cancellation by a Kelvin wave of triangular wave form (which, if one so desires, may be regarded as a sum of sinusoidal waves). The Kelvin wave travels along the side walls in a cyclonic sense and produces initially (i.e., at time \( b/f R_2 > t > 0 \) a nonzero thermocline elevation, constant for a substantial length of the side wall. The magnitude of this thermocline elevation is approximately (near the left-hand side wall, \( x = -a/2 \) and to zeroth order in \( a \)):

\[
\zeta' = \frac{Fr}{f R_2 h + h'} \exp\left(\frac{-(x + a/2)}{R_2}\right)
\]

(21)

with corresponding surface layer velocities

\[
u = \frac{F}{f} \frac{h'}{h + h'} \left[ 1 - \exp\left(\frac{-(x + a/2)}{R_2}\right) \right]
\]

\[
v = \frac{Fr}{h + h'} \exp\left(\frac{-(x + a/2)}{R_2}\right)
\]

(21a)

These are similar to the relationships that apply to the nearshore region in the simple coastal jet model. However, the combination of the surface and internal modes now leads to a different result in the bottom layer. The total transport being zero everywhere, return flow occurs below the thermocline with velocities \( v' = -v(h/h'), u' = -u(h/h') \). The momentum is imparted to the bottom layer in this case.

Fig. 4. Combination of static thermocline elevation pattern associated with flow pattern of Fig. 3 ("setdown") with slowly traveling triangular Kelvin wave, composed of sum of sinusoidal components. Diagram shows thermocline elevation distribution along longer shore to the right of the wind.
by the setup (longshore pressure gradient), which opposes the wind stress. The same pressure gradient reduces the longshore velocity of the surface layer by the factor \( \frac{h'}{h + h'} \), as compared to the simple coastal jet model without setup. Outside the coastal boundary layers of scale width \( R_2 \) there is again no flow and little thermocline displacement (more precisely, \( \zeta' \) is of order \( \zeta \)), and a surface elevation gradient balances the wind stress.

We may sum up this discussion of a finite, constant-depth basin in the observation that the longshore pressure gradient produced by the end walls establishes return flow below the thermocline within the coastal boundary layer, where along an infinite shore the water would remain stagnant. There is also another effect: the slow propagation around the basin of the stagnation point, in a cyclonic sense (Csanady and Scott, 1974). This follows from the propagation of the Kelvin waves we already mentioned but is outside the scope of the coastal jet conceptual model.

5. Depth Variations

Next we would like to extend the coastal jet concept to still more realistic models by allowing the depth of the basin to vary. To the accuracy of linearized theory, and of the hydrostatic approximation, equations 1 remain valid if \( h \) and \( h' \) depend on \( x \) and \( y \), except of course where \( h \) or \( h' \) tend to zero (in the linearization process \( h \) and \( h' \) are assumed large compared to \( \zeta \) and \( \zeta' \)). The coastal jet concept is only useful when there is a long and straight coast, so that the most general model we need consider is a long and narrow basin with arbitrary depth distribution in the cross section, as illustrated in Fig. 5. The basin has end walls, but these are supposed far away and the depth is only a function of the \( x \)-coordinate. Where \( h \) or \( h' \) tends to zero, equations 1 have singularities that lead to some physically unrealistic features of the solutions. These do not prove to be unduly disturbing (the physical reasons for the anomalies being clear enough) so that we shall not concern ourselves with the corrections that, strictly speaking, would have to be made near these singularities.

---

Fig. 5. Long and narrow basin model with arbitrary depth distribution.
A thermocline is present only where the total depth is greater than the top layer equilibrium depth \( h_0 \). Within this part of the basin all six of equations 1 must be used, outside the \( h + h' = h_0 \) locus only the first three (but setting \( \zeta' = 0 \)). Within the two-layer portion, and sufficiently far from the end walls, a coastal jet type of response may be expected to be described by a solution of equations 1 of the following form:

\[
\begin{align*}
U &= U(x) & U' &= U'(x) \\
V &= B(x)t & V' &= B'(x)t \\
\zeta &= W(x)t + Sy & \zeta' &= W'(x)t + S'y
\end{align*}
\] (22)

Here \( B \) and \( B' \) are depth-integrated longshore accelerations and \( W \) and \( W' \) are vertical velocities, all functions of \( x \) only. Longshore gradients of surface elevation, \( S \), and of thermocline elevation \( S' \), are supposed constant and they represent the only influence we allow the end walls. The magnitudes of these longshore gradients may be thought to be determined by nondimensional basin dimensions, length divided by appropriate radii of deformation, as suggested by our earlier results.

Substituted into equations 1, equations 22 lead to the relationships:

\[
W = - \frac{d}{dx}(U + U')
\]

\[
B = \frac{gh}{f} \frac{dW}{dx} = - \frac{gh}{f} \frac{d^2}{dx^2}(U + U')
\]

\[
W' = - \frac{dU'}{dx}
\] (23)

\[
B' = \frac{gh'}{f} \frac{dW}{dx} + \frac{geh'}{f} \frac{d}{dx}(W' - W)
\]

\[
= \frac{gh'}{f} \frac{d^2}{dx^2}(U + U') + \frac{geh'}{f} \frac{d^2U}{dx^2}
\]

These allow longshore accelerations and vertical velocities to be calculated once the distribution of onshore–offshore transports \( U \) and \( U' \) is known. The latter are subject to (still from equations 1 and 22, eliminating \( W, B, W' \), and \( B' \) with the aid of equations 23):

\[
- \frac{gh}{f} \frac{d^2}{dx^2}(U + U') + fU = -ghS + F
\]

\[
- \frac{gh'}{f} \frac{d^2}{dx^2}(U + U') + \frac{geh'}{f} \frac{d^2U}{dx^2} + fU' = -gh'S - geh'(S' - S)
\] (24)

Our previous results lead us to suspect that the latter two equations have solutions consisting of a sum of "surface" and "internal" modes. For small proportionate density defect \( \varepsilon \) these models should be characterized by (1) a vertically uniform velocity, and (2) a vanishing total transport. We therefore seek solutions consisting of two additive components, indexed \( 1 \) and 2, each subject to constraints as follows:

\[
U = U_1 + U_2 \quad U' = U'_1 + U'_2
\]

\[
\frac{U'_1}{h'} - \frac{U'_1}{h} = 0 \left( \frac{U_1}{h_0} \right)
\]

\[
U_2 + U'_2 = 0(\varepsilon U_2)
\] (25)
We also anticipate that the two solutions will be scaled by very different characteristic lengths, so that when equations 25 are substituted into equations 24, index 1 and index 2 quantities should balance separately. Dividing the first of equations 24 by \( h \), the second by \( h' \), and subtracting, we find

\[
- \frac{g e}{f} \frac{d^2 U}{dx^2} + f \left( \frac{U}{h} - \frac{U'}{h'} \right) = \frac{F}{h} + g e (S' - S) \quad (24a)
\]

Substituting now the form of solution postulated in equations 25 we obtain

\[
- \frac{g e}{f} \frac{d^2 U_2}{dx^2} + f \left( \frac{1}{h} + \frac{1}{h'} \right) U_2 = \frac{F}{h} + g e (S' - S) \\
+ \left[ \left( \frac{f U_1'}{h'} - \frac{f U_1}{h} \right) + \frac{g e d^2 U_1}{f dx^2} - \frac{U_2 + U_2'}{h'} \right] \quad (26)
\]

By hypothesis the last square-bracketed term on the right of this equation is of order \( \varepsilon \). The second term on the right is also of order \( \varepsilon \), unless either \( S \) or \( S' \) is much larger than \( F/gh \), a possibility we disallow far from the end walls. The first term on the left also contains \( \varepsilon \), but it is not negligible if (as expected) the length scale or variations of \( U_2 \) is of order \((ge h_0)^{1/2} f^{-1}\). Dropping the order \( \varepsilon \) terms from equation 26 we have after some cross multiplications:

\[
\frac{d^2 U_2}{dx^2} - \frac{f^2}{g e} \left( \frac{1}{h} + \frac{1}{h'} \right) U_2 = - \frac{f F}{g e h} \quad (27)
\]

Within the two-layer portion of our system \( h = h_0 = \text{constant} \), but \( h' \) is variable. Equation 27 then constitutes a relatively simple second-order equation from which \( U_2 \) may be determined. For \( h' \) also constant the solution is exactly what we already found for the internal mode in a constant-depth canal. Boundary conditions are \( U_2 = 0 \approx -U_2 \) at both locations where the thermocline intersects the bottom.

Adding the two equations 24 leads to

\[
\frac{g}{f} (h + h') \frac{d^2}{dx^2} (U + U') - f (U + U') = g (h + h') S - F + g e h' (S' - S) + \frac{1}{f} \frac{d^2 U}{dx^2} \quad (24b)
\]

Substituting the "Ansatz" of equation 25 and dropping undifferentiated terms multiplied by \( \varepsilon \) there is left:

\[
\frac{g}{f} (h + h') \frac{d^2}{dx^2} (U_1 + U_1') - f (U_1 + U_1') = g (h + h') S - F + g e h' \frac{d^2 U_2}{f dx^2} \\
- \frac{g}{f} (h + h') \frac{d^2}{dx^2} (U_2 + U_2') \quad (28)
\]

Although \( U_2 + U_2' \) is by hypothesis of order \( \varepsilon U_2 \), the last term on the right is not negligible because its horizontal variation is scaled by \((ge h_0)^{1/2} f^{-1}\), as is the variation of the second-last term, both according to equation 27. Because the other terms in the equation are scaled differently, these two terms must balance between themselves, a condition from which the precise variation of \((U_2 + U_2')\) may be calculated. This is an
order $\varepsilon$ quantity of no special interest and we shall not further concern ourselves with it. The remaining terms in the equation leave

$$\frac{d^2}{dx^2} (U_1 + U_1') - \frac{f^2}{g(h + h')} (U_1 + U_1') = fS - \frac{fF}{g(h + h')}$$  \hspace{1cm} (29)

The same equation applies outside the two-layer portion, but of course with $h' = 0$, $U' = 0$.

The solution of this equation yields the response in the surface or barotropic mode, which is (to order one) independent of stratification. With constant $h + h'$ our earlier results are again recovered. As one easily demonstrates, equations 27 and 29 possess solutions for at least some well behaved $h'$ distributions, so that they constitute the extension of the coastal jet model to the variable depth case, the conjectures incorporated in equations 22 and 25 having been substantiated.

### 6. The Pattern of Barotropic Flow

Equation 29 describes the aperiodic response of a variable-depth basin in the surface or barotropic mode to suddenly impose longshore wind. As we have already pointed out, we expect the coastal jet conceptual model to be useful along a long, straight coast; hence we concentrate our attention on a basin much longer than it is wide, illustrated schematically in Fig. 5.

A scale analysis of the problem shows the second term on the left of equation 29 to be small, if the basin width is small compared to an appropriately defined radius of deformation. Let the maximum depth of the basin be $H$, and let us nondimensionalize the terms of equation 29 according to the following scheme:

$$R^2 = \frac{gH}{f^2}$$

$$x^* = \frac{x}{R}, \quad h^* = \frac{h + h'}{H}$$  \hspace{1cm} (30)

$$U^* = \frac{f(U_1 + U_1')}{F}, \quad S^* = \frac{gHS}{F}$$

Let the shores of the basin be at $x = -a_1, a_2$, and assume that

$$a_{1,2}^* = \frac{a_{1,2}}{R} \ll 1$$  \hspace{1cm} (31)

Equation 29 becomes with the nondimensional quantities introduced:

$$\frac{d^2U^*}{dx^{*2}} = \frac{U^*}{h^*} + S^* - \frac{1}{h^*}$$  \hspace{1cm} (29a)

The nondimensional cross flow $U^*$ is defined as the fraction of Ekman transport and we may expect this to be at most of order unity. The nondimensional pressure gradient $S^*$ is equal to one if the pressure gradient exactly balances the wind stress in the deepest part of the basin. We may expect this to be also of order unity; in a sufficiently long basin it could even be of smaller order, according to our previous results. Thus unless the singularity at the shore (where $h^*$ tends to zero) causes difficulty, $U^*$ obtained by a double integration of equation 29a should be of order $a_{1,2}^*$, that is, small. An ordinary sloping beach, where $h^*$ varies linearly with $(x \pm a_{1,2})$
does not cause a problem and the conclusion regarding the order of magnitude of $U^*$ stands, unless the bottom slope becomes unrealistically small.

One may further verify the truth of this assertion using some specific model. Figure 6 illustrates an idealized cross section with simple, symmetrical sloping beaches ($a_1 = a_2 = a$) and a constant-depth center. The dimensions of this cross section approximate those of Lake Ontario. Over the sloping parts the solution of equation 29a is

$$U^* = C_1 \lambda I_1(\lambda) - \lambda K_1(\lambda) - S^* h^* + 1$$  \hspace{1cm} (32)

where $I_1(\lambda), K_1(\lambda)$ are modified Bessel functions, $C_1$ is an integration constant, and

$$\lambda = \sqrt{\frac{4|x + a|H}{sR}}$$

with $s$ the bottom slope $s = d(h + h')/dx$. Over the center part, on the other hand:

$$U^* = 1 - S^* + C_2 \cosh x^*$$  \hspace{1cm} (33)

with $C_2$ another integration constant. These solutions already satisfy one boundary condition each, zero offshore flow at the shore, and symmetry of the $U^*$ distribution. Matching $U^*$ and $dU^*/dx$ (i.e., transport and surface elevation) at the point where the two sections join, one may determine the constants. Given the quantitative data of Fig. 6, the center (maximum) value of $U^*$ turns out to be of order $10^{-3}$. The dimensional quantity $U_1 + U'_1$ is thus three orders of magnitude less than the Ekman flux $F/f$.

The depth-integrated longshore acceleration in the surface mode is, using equations 23 and 29:

$$B_1 + B'_1 = -\frac{g}{f}(h + h') \frac{d^2}{dx^2} (U_1 + U'_1)$$

$$= -g(h + h')S + F - f(U_1 + U'_1)$$  \hspace{1cm} (34)

As we have just seen, the last term in this equation is negligible, so that the longshore momentum balance (which equation 34 represents) is between wind stress, pressure gradient, and acceleration, Coriolis force due to cross flow being of subordinate importance.

Fig. 6. Idealized model of Lake Ontario cross section.
With \( F \) and \( S \) constant, one term on the right of equation 34 is constant and the other varies as \( h + h' \), so that the acceleration \( B_1 + B'_1 \) also varies. In the constant-depth case, for a sufficiently short basin, we have seen that \( B_1 + B'_1 \) was negligible everywhere and the wind stress was balanced by the pressure gradient. The nearest analogue in the present case is that the cross-sectional integrals of the two balance, that is,

\[
gS \int_{-a_1}^{a_2} (h + h') \, dx = F(a_2 + a_1)
\]

(35)

This also implies then:

\[
\int_{-a_1}^{a_2} (B_1 + B'_1) \, dx = 0
\]

(36)

so that no fluid accumulates windward of the cross section considered.

It is tempting to impose equation 35 (or equivalently, 36) as a condition and call it part of the coastal jet conceptual model for an appropriately small basin. However, in order to maintain reasonable correspondence with what might be observable coastal jets, we would have to include all low-frequency modes of motion that contribute to coastal currents. To the extent that the effects of the Coriolis force are entirely negligible, it is easy enough to justify equation 36, because any net accelerations are then associated with the relatively high seiche frequencies. In a rotating basin of variable depth, on the other hand, there may exist "second class motions" (Ball, 1965), analogous to "shelf waves" or "topographic Rossby waves." The combination of these waves with the steady-state flow pattern may give rise to a completely different resultant, much as the static elevation plus Kelvin waves produce a coastal jet in a constant-depth basin unlike either of its constituents. This question needs further mathematical investigation (and, of course, corresponding further scrutiny of experimental evidence). For the time being, we accept equations 35 and 36 on an intuitive basis, as probably reasonable for describing the aperiodic barotropic flow in a variable-depth basin of modest size, at least for a period after a wind-stress impulse, before the Coriolis force can develop some more complex effects. [For further remarks on this question see a recent discussion by Simons (1974) and my reply.]

Bennett (1974) seems to have been the first to explore the interesting implications of equation 34, with equation 36 imposed as a condition. The integral on the left of equation 35 is the total fluid-filled area of the canal cross section, which we may use to define a mean depth:

\[
h_m = \frac{1}{a_2 + a_1} \int_{-a_1}^{a_2} (h + h') \, dx
\]

(37)

The longshore pressure gradient is then \( S = F/gh_m \), and equation 34 gives zero depth-integrated acceleration \( B_1 + B'_1 \) for those points in the cross section where \( h + h' = h_m \). In shallower water the wind stress overcomes the pressure gradient and accelerates the water downwind. Indeed in very shallow water, where \( h + h' \ll h_m \), the effect of the pressure gradient becomes negligible. This should hold independently of the truth of the intuitively imposed equation 36, as long as the pressure gradient \( S \) is of order \( F/gh_m \). In water deeper than \( h_m \) the pressure gradient overwhelms the wind stress and the developing volume transport is against the wind.
We have arrived at these results for a given basin cross section. Their implication for a long and narrow basin is a sort of double-gyre flow pattern, illustrated in Fig. 7, the streamlines shown indicating depth-integrated volume transport. The velocities are the transports divided by water depth, that is, relatively large in shallow water. We conclude that a pronounced barotropic coastal jet should be present in a variable-depth basin outside the mean depth contour, a very different result from the constant-depth case. In water much shallower than the cross-sectional mean depth $h_m$ the momentum gain is nearly equal to the wind stress impulse divided by total depth. This is exactly the same result as we found earlier for an infinite coast (without end walls), the reason for the correspondence being that the effects of longshore gradients in shallow water are unimportant.

7. Baroclinic Flow Over Variable Depth

We return now to equation 27 and inquire how internal mode coastal jets are affected by depth variations. This equation of course applies only where the bottom layer equilibrium depth $h'$ is nonzero and where the top layer depth $h = h_0$ is constant. A solution of equation 27 yields the top layer baroclinic onshore-offshore transport $U_2$, the bottom layer transport, $U'_2$, being equal and opposite to order $\varepsilon$. Other quantities of interest may then be found from equations 23, especially thermocline movement and longshore acceleration:

\[ W'_2 = \frac{dU_2}{dx} \]

\[ B_2 = -\frac{gehh'}{f(h + h')} \frac{d^2U_2}{dx^2} \]

and $B'_2 = -B_2$. The surface elevation (or rather its rate of movement, $W$) is a quantity of order $\varepsilon^{1/2}$.

Equation 27 is a second-order nonhomogeneous equation of relatively simple structure which, for simple $h'$ distributions, reduces to the hypergeometric equation. Specifically for linearly varying $h'$ ($h' = xs$, with $s$ the bottom slope) the equation transforms into

\[ \frac{d^2U_2}{dx^2} - \left(\frac{1}{4} + \frac{k}{x_*}\right)U_* = -\frac{1}{4} \]
where

\[ U_\ast = \frac{fU_2}{F} \]

\[ x_\ast = \frac{2x}{L} \]

\[ k = \frac{h_0}{2sL} \]

with

\[ L = \frac{(geh_0)^{1/2}}{f} \]

This is Whittaker's equation with a constant nonhomogeneous term, the solutions of which are generalized Struve functions (Babister, 1967). The solutions reduce to a simpler form for special values of the slope parameter \( k \). The character of the solution may be illustrated using \( k = 0.5 \), in which case one finds (Csanady, 1971):

\[ U_2 = C \frac{Fx}{fL} \left[ I_0 \left( \frac{x}{L} \right) + I_1 \left( \frac{x}{L} \right) \right] - \frac{Fx}{fL} \left[ 1 + \frac{\pi}{2} L_0 \left( \frac{x}{L} \right) + \frac{\pi}{2} L_1 \left( \frac{x}{2} \right) \right] \]

(40)

where \( I_0 \) and \( I_1 \) are modified Bessel functions \( L_0 \) and \( L_1 \) are modified Struve functions, and \( C \) is an integration constant. The solution already satisfies the boundary condition of zero normal transport at the coast. The constant \( C \) may be determined by patching the solution 40 to a constant-depth model at some specific \( x = x_0 \), with continuous \( U_2 \) and \( dU_2/dx \).

In a quantitative illustration the following "typical" values may be used:

\[ s = 3.10^{-3} \]
\[ \varepsilon = 2.10^{-3} \]
\[ h_0 = 18 \text{ m} \]
\[ f = 10^{-4}/\text{sec} \]

These result in \( k = 0.5, L = 6 \text{ km} \). Transition to constant depth may be supposed to occur at \( x = 30 \text{ km} \), resulting in a maximum bottom layer depth of 90 m, total water depth of 108 m. All these data are representative of conditions in shallow seas, especially in the Great Lakes or over the eastern seaboard continental shelf, although the slope chosen is on the low side.

When the transition to constant depth takes place at a distance much larger than \( L \), the value of the constant \( C \) becomes approximately \( \pi/2 \) and \( U_2 \) tends with large \( x/L \) to the constant value \( F/f \) (the total Ekman transport) already over the sloping part of the bottom. In this special case the depth-integrated longshore acceleration is

\[ B_2 = F \frac{x}{L} \left( 1 + \frac{x}{L} \right)^{-1} + 1 - \frac{\pi}{2} \left[ I_0 \left( \frac{x}{L} \right) - L_0 \left( \frac{x}{L} \right) \right] - \frac{\pi}{2} \left[ I_1 \left( \frac{x}{L} \right) - L_1 \left( \frac{x}{L} \right) \right] \]

(41)

This vanishes at \( x/L = 0 \) and \( x/L \to \infty \), the variation in between being illustrated in Fig. 8. Notable is the relatively small maximum value of \( B_2/F \); in a constant-depth model the maximum of this quantity is \( h'(h + h') \) (see equation 9), which is usually close to unity. In the present example, the maximum \( B_2 \) occurs where \( h'(h + h') \) is
approximately $\frac{1}{2}$, but by the time this depth is reached, the distance from the thermocline-bottom intersection is of order $L$, so that the baroclinic acceleration has decayed appreciably.

The top layer longshore acceleration is $B_2$ divided by the constant top layer depth $h_0$ and therefore also varies as $B_2/F$ (Fig. 8). However, the bottom layer longshore acceleration is different, $-B_2/sx$, which has the nonzero value of $-(F/h_0)(2 - \pi/2)$ at the shore.

The upward velocity of the thermocline may be calculated to be, from equations 38 and 40, with $C = \pi/2$:

$$W_2 = \frac{F}{fL} \left\{ \frac{\pi}{2} \left( 1 + \frac{x}{L} \right) \left[ I_0 \left( \frac{x}{L} \right) - L_0 \left( \frac{x}{L} \right) \right] + \frac{\pi x}{2L} \left[ I_1 \left( \frac{x}{L} \right) - L_1 \left( \frac{x}{L} \right) \right] - \left( 1 + \frac{x}{L} \right) \right\}$$

(42)

This also vanishes at large $x/L$, while its value at $x/L = 0$ is $(\pi/2 - 1)F/fL$, or a little over half of the value which applies in a constant-depth model with a relatively large $h/L$ ratio (see again equation 9).

We may now combine the barotropic and baroclinic components of the flow to obtain the total response over a sloping beach. We assume for simplicity that the length scale of baroclinic jets, $L$, is small compared to basin dimensions, and that even the beach of constant slope $s$ extends much further from shore than $L$. This also implies...
that the depth at a distance of order $L$ from the thermocline–bottom intersection is small compared to basin average depth, so that the effect of any longshore pressure gradient on the barotropic flow is small within this same range.

The longshore barotropic acceleration is then (by equation 34) $F/(h + h')$. At the point where the thermocline intersects the bottom this gives longshore velocities $v = Ft/h_0$, $v' = (\pi/2 - 1)Ft/h_0$ (the latter with the results of the above typical example). The depth of the bottom layer is here of course zero, so that the wind-induced momentum is all in the top layer. The bottom layer, however, is not at rest now (in contrast with the constant-depth model, at the shore). The physical reason for the difference is easily discovered if we recall that the thermocline is rising in our typical beach model with a velocity of $W' = (\pi/2 - 1)F/f L$. Because of the presence of a bottom slope, this requires an inflow velocity $u' = U'/h'$ in the bottom layer of a magnitude equal to vertical velocity divided by the slope, $u' = -W'/s = -(\pi/2 - 1)F/f h_0$ (noting that in our typical example with $k = 0.5$, $L = h_0/s$). The Coriolis force acting on an inward moving parcel of fluid then produces the longshore acceleration $fu' = (\pi/2 - 1)F/h_0$, explaining our earlier result.

At distances long compared to $L$ the baroclinic contribution to $B$ and $B'$ decays to zero and the longshore acceleration pattern is what we described in the preceding section, evenly distributed over the total depth. There is, however, equal and opposite onshore–offshore flow in the top and bottom layers. The momentum of the wind stress is transferred to the bottom layer by the medium of the Coriolis force, as we discussed in greater detail in connection with the constant-depth model.

To sum up this discussion of depth effects, we may conclude that depth variations have an important influence on the pattern of barotropic flow basinwide (this may be pursued beyond the scope of the coastal jet conceptual model; see, e.g., Csanady 1973, 1974). In shallow water the net effect is that the momentum imparted by the wind simply accelerates the water column, the longshore pressure gradient having a negligible influence. In this respect, the variable-depth shore zone behaves as an infinitely long coast in the simplest coastal jet model. The baroclinic contribution still rearranges the total momentum input in favor of the top layer, within a nearshore band of scale width $L$, which is very nearly the same as the baroclinic radius of deformation $R_2$ in a constant-depth basin of large bottom layer to top layer depth ratio. However, the monopoly of the top layer on the windward momentum is not complete over a sloping beach, because the bottom layer acquires windward velocity even very close to the thermocline–bottom intersection. In our “typical” example (which was actually closer to the flattest beach extreme) the bottom layer velocity near $h' = 0$ was about half the top layer one in the same location. A larger bottom slope would tend to reduce this toward the zero value appropriate for a vertical beach. The top layer longshore acceleration is influenced relatively little by the baroclinic contribution for two reasons: (1) this contribution decays with scale length $L$, and (2) the bottom layer grows only slowly in depth within this distance range, and can absorb only small amounts of wind-imparted momentum. The net result, however, is still that within a band extending a distance of order $L$ beyond the thermocline–bottom intersection the momentum input of the wind is mainly reflected in a momentum gain by the top layer.

It is anticlimactic to arrive at the result that the simplest, infinite shore, constant-depth coastal jet model predicted much the same behavior as the variable-depth, closed basin one (albeit a long and narrow basin). The differences in detail are not as important practically as the similarities, although they are essential from the point of view of a dynamical understanding of the phenomena involved.
8. Application to Observed Coastal Currents

During and in preparation for the International Field Year on the Great Lakes (IFYGL, carried out during 1972) a systematic series of observations have been carried out on the coastal boundary layer of Lake Ontario, defined as the band extending from 0 to about 10 km from shore. Lake Ontario is much longer than it is wide (by about a factor of 5) and it possesses some relatively straight, uncomplicated shorelines (Fig. 9). Both westerly and easterly storms are moderately frequent, and are sometimes preceded and followed by quiescent periods, allowing a study of isolated longshore wind-stress impulses. During the summer months a resonably sharp thermocline separates the top epilimmion from a colder and slightly denser hypolimmion, producing conditions to which the coastal jet conceptual model should apply. The IFYGL observations have indeed produced flow phenomena very much with characteristics as discussed above; some of these are illustrated in Figs. 10–12b.

All these illustrations show examples of coastal jets in their initial development, that is, more or less immediately following a shore-parallel wind stress impulse acting upon a relatively quiescent lake. Their predominant features are as follows:

1. High velocities in the jets are confined to the warm layer and to a nearshore band of the order of 5 km width. Water below the thermocline moves slowly (with a velocity of a few cm/sec) in the same direction as above.

2. A thermocline upwelling or downwelling occurs in the region of the jet. A downwelling accompanies flow leaving the shore to the right, upwelling an opposite current. The velocity of the warm water relative to the cold is geostrophically balanced by the thermocline slope, to a good approximation.

3. During IFYGL the wind-stress impulse $F_I$ could be estimated from over-the-lake wind observations. The total depth-integrated momentum at the core of the jet

![Fig. 9. L. Ontario with locations of coastal current studies.](image-url)
Fig. 10. Example of coastal jet, 23 July 1972, south shore showing isotherms (°C, left) and constant longshore velocity contours (cm sec\(^{-1}\)). Direction of flow is such as to leave shore to the right.

was always of the same order of magnitude as the wind-stress impulse (although somewhat less, typically 70% of \(F_t\)).

4. The distribution of depth-integrated momentum with distance from shore (Fig. 11) showed a peak at 6–7 km from shore, presumably owing to strong frictional influences in water less than 30 m deep. In somewhat deeper water the depth-integrated momentum was close to the wind-stress impulse (as already pointed out) but it then decreased to zero roughly where the depth became equal to the cross-sectional average.
On the basis of this evidence it appears legitimate to regard flow structures of the kind depicted in Figs. 10–12 as distinct observable phenomena, the principal features of which are in good accord with the coastal jet conceptual model. There are other characteristics of these nearshore currents which are more complex: for instance, the reversal of flow upon the passage of a Kelvin wave (Csanady and Scott, 1974), or the observed preference for a flow direction leaving the shore to the right (Blanton, 1974),

![Diagram](image)

Fig. 11. (a) isotherms, °C; (b) constant longshore velocity contours at Oshawa immediately after major storm (10 Oct. 1972). Flow now leaves shore to the left.
not to speak of the effects of friction clearly evident in Fig. 11. Some of these complexities may be accounted for by an extension of the coastal jet conceptual model, whereas others will no doubt require recourse to different conceptual models. It seems appropriate at this point to let the case rest.

In conclusion, a few words may be said regarding the application of the same conceptual model to flow phenomena over the continental shelves. Given the depths and widths of these, there can be little doubt that very much the same phenomena as documented in detail on the Great Lakes will occur within 10 km or so of oceanic coasts. We have carried out a pilot study of coastal currents south of Long Island (so far unpublished) which confirms this expectation. Fragmented reports of similar phenomena have surfaced from time to time (see, e.g., the review of Bumpus, 1973, referring to upwelling along the Maine coast) but have not been systematically assembled. In studying upwelling along the Oregon coast, O'Brien and his collaborators have explicitly invoked the coastal jet conceptual model (see O'Brien and Hurlburt, 1972; Thompson and O'Brien, 1973). It appears, however, that the Oregon upwelling may be affected by some nonlocal influences, which produce, among other effects, a poleward undercurrent. Possibly the coastal jet conceptual model will prove useful
in describing short-term response phenomena in this situation, becoming one of several tools in a larger kit to which one may have to have recourse to understand fully a more complex total picture.

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References


