1 Introduction

Waves are generated by a restoring force to a disturbance

Examples of waves: sound waves (due to pressure disturbances), tides (caused by moon/sun gravitation), ocean waves

Ocean waves are generally a disturbance due to wind and a restoring force from gravity

Think of them on the surface of the ocean, but can be within the water column as well (along density surfaces)
Figure 1: Wave length and wave period are shown, dependent on what is represented by the x-axis. Amplitude is also shown. Snapshot from wave illustration tool made by Rob Hetland.

- Waves coming onto the beach, waves in the open ocean, wave ripples moving away from a rock dropped into water

2 Linear Theory

Waves are complicated, so let’s start with some assumptions to narrow our scope to the basics:

- Amplitude of waves on water surface is small
- Flow is 2D
- Waves travel in one direction only
- No rotation or viscosity

Form for sea surface elevation, $\zeta$, of a wave in the $x$-direction is

$$\zeta = a \sin (kx - \omega t),$$

(1)

- $\omega$ wave frequency in radians/s
  $\omega = 2\pi f = \frac{2\pi}{T}$, $T$ wave period
  $T$ is the time between successive waves to pass
- $k$ wave number in 1/m
  $k = \frac{2\pi}{L}$, $L$ wavelength
  $L$ is the distance between two waves for a fixed time

Various wave properties are illustrated in Figure 1. The form of a wave (Equation 1) tells us that a wave can be approximately described as a constant amplitude multiplied by a sinusoidal function, which varies in time between -1 and 1.
2.1 Dispersion Relation

The dispersion relation tells us how $\omega$ and $k$ depend on each other. Given our assumptions, the dispersion relation is

$$\omega^2 = gk \tanh kd, \quad (2)$$

where $d$ is the depth of the water and $g$ is the gravitational constant (9.81 $m/s^2$). See Figure 2 for an illustration.

Taking extreme values of parameters allows us to simplify and examine the important features of specific cases. Here we look at two physical cases: when the depth $d$ is much greater or much less than the wavelength $L$ of a wave.

Deep water ($d \gg L$ or $d/L \gg 1$):

$$\frac{2\pi d}{L} \to \infty \text{ as } d/L \to \infty \quad (3)$$

$$\Rightarrow \tanh (kd) = \tanh \left( \frac{2\pi d}{L} \right) \to 1 \text{ as } d/L \to \infty \quad (4)$$

$$\Rightarrow \omega^2 \approx gk. \quad (5)$$

You can check the behavior of $\tanh()$ by plotting or trying different numbers.

Shallow water ($d \ll L$ or $d/L \ll 1$):

$$\tanh \left( \frac{2\pi d}{L} \right) \to \frac{2\pi d}{L} \text{ as } d/L \to 0 \quad (6)$$

$$\Rightarrow \omega^2 \approx gk^2d \quad (7)$$
2.2 Phase and Group Speed

Phase speed $c$ is that speed that a particular wave travels at; it is the wave movement that we can most readily see. It can be derived as:

\[ \text{velocity} = \frac{\text{distance}}{\text{time}} \]

\[ c = \frac{L}{T} \]  \hspace{2cm} (8)

\[ = \frac{2\pi}{k} \frac{\omega}{2\pi} \]  \hspace{2cm} (9)

\[ = \frac{\omega}{k}. \]  \hspace{2cm} (10)

We can use again learn about the phase speed by looking at special cases. Deep water:

\[ c = \frac{\omega}{k} \]  \hspace{2cm} (12)

\[ = \sqrt{gk} \]  \hspace{2cm} (13)

\[ = \sqrt{g} \]  \hspace{2cm} (14)

\[ = \frac{g}{\omega}. \]  \hspace{2cm} (15)

Shallow water:

\[ c = \frac{\omega}{k} \]  \hspace{2cm} (16)

\[ = \sqrt{gk^2d} \]  \hspace{2cm} (17)

\[ = \sqrt{gd}. \]  \hspace{2cm} (18)

The important thing to note here is that in the deep water case, the phase speed is a function of the wave number/frequency. This means that waves of different wave length will travel at different speeds. These waves are then called dispersive. Oppositely, the phase speed for shallow water waves does not depend on the wave properties, only gravity and the depth of the water, and waves will travel at the same speed. These waves are nondispersive.

Group speed is the rate of travel of a packet of waves; it is the speed at which energy and information propagate. It is defined as

\[ c_g = \frac{\partial \omega}{\partial k}. \]  \hspace{2cm} (19)
Deep water:

\[ c_g = \frac{\partial}{\partial k} \left( (gk)^{1/2} \right) \]  
\[ = \frac{1}{2} (gk)^{-1/2} g \]  
\[ = \frac{1}{2} \sqrt{g/k} \]  
\[ = \frac{1}{2} c \]

Shallow water:

\[ c_g = \frac{\partial}{\partial k} \left( (g k^2 d)^{1/2} \right) \]  
\[ = \frac{1}{2} (g k^2 d)^{-1/2} 2gkd \]  
\[ = \sqrt{gd} \]  
\[ = c \]

For shallow water waves, the group and phase speeds are equal. For deep water waves, the group speed is half the phase speed. This means that individual waves travel faster than groups of waves. This effect is possible because deep water waves are dispersive, and the phase speed depends on the wave properties.

In general, examples of the different scenarios possible between two waves interacting can be found using Rob’s twoWave simulator. Different scenarios include:

- \( c > c_g, c_g > 0 \) : \( \omega_1 = 0.5, \omega_2 = 0.44, k_1 = 1.72, k_2 = 1.44 \)
- \( c < c_g, c_g > 0 \) : \( \omega_1 = 0.5, \omega_2 = 0.39, k_1 = 1.72, k_2 = 1.44 \)
- \( c > c_g, c_g < 0 \) : \( \omega_1 = 0.5, \omega_2 = 0.72, k_1 = 1.72, k_2 = 1.44 \)

Thus, the wave frequency/number can be adjusted for the simulated waves to cause the phase speed to be greater than or less than the group speed, and the group speed can be positive or negative.

Examples can also be found online. In this case, the waves are approximately moving in just one direction (outward from the boat track), as we assumed. It is hard to see the packet of waves moving, but we can see individual waves appear from the tail end and move forward, as in the model case.

### 3 Wave Representation

In order to study waves and the surface of the sea, we need to be able to represent the surface. For waves, this is typically done using an energy spectrum, which is based on the concept of Fourier series.
4
3
2
1
0
1
2
3
4
Fourier series of $f(x)$ with various numbers of terms included

$f(x) = x$

Figure 3: Different numbers of terms in the Fourier series representation of the function $f(x)$ have been included and overlaid. Example from O’Neil (2003)

3.1 Fourier Series

Fourier series are summations of an infinite number of terms made up of sine and cosines functions, unlike Taylor series, which are made up of polynomial terms. Fourier series can be used to represent a surface on a particular interval of space or time. They are a commonly-used tool for information representation and manipulation.

For example, let us consider the function $f(x) = x$ for $-\pi \leq x \leq \pi$. This can be represented using a Fourier series for the interval $-\pi < x < \pi$ as

$$f(x) = x = \sum_{n=1}^{\infty} \frac{2}{n} (-1)^{n+1} \sin(nx).$$

The function actually equals the summation when it includes an infinite number of terms. However, in practice we select a finite number of terms to approximate our function. How many terms we include will affect how good of a representation it is.

Figure 3 shows a plot of $f(x) = x$ overlaid with multiple representations of the function as Fourier series with differing numbers of included terms from the series. Clearly, as more terms are included, the approximation to the function is improved, and it will be exact when an infinite number of terms are included. Note that the endpoints are never correct, but $-\pi$ and $\pi$ are not included in the interval of representation for the Fourier series. This example was made using Python script `fourier.py` and is available if anyone is interested.
Wave Energy

Wave energy \( E \) in Joules per square meter is related to the variance of sea-surface displacement \( \zeta \) by:

\[
E = \rho w g \langle \zeta^2 \rangle
\]

where \( \rho \) is water density, \( g \) is gravity, and the brackets denote a time or space average.

**Significant Wave Height**

What do we mean by wave height? If we look at wind-driven sea, we see waves of various heights. Some are much larger than most, others are much smaller (Figure 16.2). A practical definition that is often used is the height of the highest \( 1/3 \) of the waves, \( H_{1/3} \).

The height is computed as follows: measure wave height for a few minutes, pick out say 20 waves and record their heights. Pick the 40 largest waves and calculate the average height of the 40 values. This is \( H_{1/3} \) for the wave record.

The concept of significant wave height was developed during the World War II as part of a project to forecast ocean wave heights and periods. Wiegel (1964: p. 198) reports that work at the Scripps Institution of Oceanography showed... wave heights estimated by observers correspond to the average of the highest 20 to 40 per cent of waves. Originally, the term significant wave height was attached to the average of these observations, the highest 30 per cent of the waves, but has evolved to become the average of the highest one-third of the waves, (designated \( H_S \) or \( H_{1/3} \)).

More recently, significant wave height is calculated from measured wave displacement. If the sea contains a narrow range of wave frequencies, \( H_{1/3} \) is related to the standard deviation of sea-surface displacement (Nas, 1963:22; Hoffman and Karst, 1975):

\[
H_{1/3} = 4\langle \zeta^2 \rangle^{1/2}
\]

where \( \langle \zeta^2 \rangle^{1/2} \) is the standard deviation of surface displacement. This relationship is much more useful, and it is now the accepted way to calculate wave height from wave measurements.

### 3.2 Wave Spectrum

The wave energy spectrum gives the distribution of the variance of sea-surface height as a function of frequency. There is a mathematical way, based on Fourier series, to move between information in time space to frequency space, to get this representation. Spectra are used as a convenient way to represent information in some applications. For example, tracking any one wave in the ocean is very difficult and not necessarily very helpful, but knowing a measure of the strength of the waves (wave height) and the frequency of the waves (which also gives period and, if you know the dispersion relation, the wave number) gives the relevant information.

Wave energy is calculated as:

\[
E = \rho g \langle \zeta^2 \rangle, \quad [J/m^2]
\]

where \( \rho \) is the water density and \( \langle \zeta^2 \rangle \) gives the variance of the sea surface displacement. Wave energy then can be represented using the wave energy spectrum.

We should specify what we mean by wave height. Two ways of calculating specific wave height are:

- As the height of the highest \( 1/3 \) of the waves, averaged together
- \( H_{1/3} = 4\langle \zeta^2 \rangle^{1/2} \)

### 3.3 Sampling Data

Let’s start with some example wave data from Stewart (2008), which is shown in Figure 4.
Figure 16.4 Sampling a 4 Hz sine wave (heavy line) every 0.2 s aliases the frequency to 1 Hz (light line). The critical frequency is $1/(2 \times 0.2) = 2.5$ Hz, which is less than 4 Hz.

Figure 5: Beware of subsampling a wave and aliasing the signal (Stewart (2008))

Aliasing is an important concept to keep in mind when collecting data. Figure 5 represents the problem: when you are trying to sample an unknown function that looks like the 4 Hz line, but you unknowingly sample too infrequently and capture what looks like the 1 Hz line. Waves with frequencies above the Nyquist critical frequency ($Ny = 1/(2\Delta)$, where $\Delta$ is the sampling frequency) cannot be captured for a given instrument setting, so instrument settings must be established such that the frequencies of interest for the application can be captured. The signal in question must also be sampled for a long enough time to capture the period of the desired wave, in order to have a full sample of the wave. These requirements can be summarized as (Stewart, 2008):

$$\frac{1}{N\Delta} < f < \frac{1}{2\Delta},$$

where $N$ is the number of samples taken and $\Delta$ is the length of the time series acquired.

The example wave data, digitized, is shown in Figure 6. This basically just points out that we only know information about the waves at a discrete number of points. This data is sampled often enough to capture the necessary wavelengths.

Two example spectra are shown in Figure 7. The spectrum, $S_n$, of $\zeta$, is shown in Figure 7(a) and is called a periodogram. Because the periodogram is noisy, we average several together, called the spectrum of the wave height, shown in Figure 7(b). This gives the distribution of the variance of sea surface height as a function of frequency. We would normally use three hours of data to compute this spectrum. Note that data in the spectra do not exceed the stated Nyquist frequency of 1.5625 Hz.
16.3. WAVES AND THE CONCEPT OF A WAVE SPECTRUM

Figure 16.3 The first 20 seconds of digitized data from figure 16.2. \( \Delta = 0.32 \text{ s} \).

Figure 6: From Stewart (2008).

Figure 16.5 The periodogram calculated from the first 164 s of data from figure 16.2. The Nyquist frequency is 1.5625 Hz.

Figure 16.6 The spectrum of waves calculated from 11 minutes of data shown in figure 7.2 by averaging four periodograms to reduce uncertainty in the spectral values. Spectral values below 0.04 Hz are in error due to noise.

Figure 7: From Stewart (2008).
3.4 Idealized Spectra

Wave energy can also be represented with idealized functional forms of spectra. Spectra for certain situations, such as wind blowing on the ocean for a long time over a large area, can be approximated using known functions, which enables quick calculations, with a possible sacrifice in accuracy depending on how realistic the idealized functions are. See Figure 8.

3.5 Application of Spectra

Stewart shows an example in his book of people using dispersion and wave spectra to track storms in the 1960s. They recorded waves for many days using three pressure gauges 60 miles west of San Diego. They calculated wave spectra for each day’s data and found the amplitudes, frequencies, and propagation direction of the low-frequency waves.

Shown in Figure 9 is wave energy as a function of both frequency and time. In order to interpret the plot, consider a distant storm generating waves. These waves will travel at different speeds due to dispersion. The wave energy will travel at the group speed, and for deep water waves, $c_g = 1/2 * g/\omega$, so that lower frequency waves will travel faster than higher frequency waves. This means that lower frequency wave energy will reach the wave gauges before higher frequency wave energy, and the lag between the arrival of the waves will align for a tell-tale signal in the resulting data. For example, the storm arriving between September 15th and 18th can be seen in the plot due to the ridgeline of wave energy.
Figure 16.1 Contours of wave energy on a frequency-time plot calculated from spectra of waves measured by pressure gauges offshore of southern California. The ridges of high wave energy show the arrival of dispersed wave trains from distant storms. The slope of the ridge is inversely proportional to distance to the storm. ∆ is distance in degrees, θ is direction of arrival of waves at California. After Munk et al. (1963).

Figure 9: From Stewart (2008).

energy, spread out in time and frequency according to the above reasoning. The angle of the ridge indicates the distance away of the storm, in degrees, and the angle of direction of the storm. This information helps to pinpoint the origin of the storm, which from this information was south of New Zealand near Antarctica.

4 Modeling

One way of finding wave information is by running a numerical model. An example of a system with wave modeling included in COAWST. Waves tend to correlate with the wind vectors in the Gulf of Mexico simulation.

5 Wave Power

Energy production from sources like this tend to be based around making something move relative to something else. The PowerBuoy is a point absorber that moves up and down in the water relative to the outer part of the structure to create energy (to be built off
the Oregon coast, Reedsport, this year!). The different segments of Pelamis move up and down at different rates to generate energy. The WaveDrageon uses overtopping to capture water, store it, and then drain it through turbines downward to generate electricity.

References


Figure 11: WaveDragon